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Walrasian objection mechanism and Mas Colell's bargaining set in economies with many commodities

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Abstract

We study the Walrasian objection mechanism in the framework of economies with a measure space of agents and a separable Banach space of commodities whose positive cone has a non-empty interior. We will provide several characterizations of Walrasian objections and use them to study the bargaining set of the economy, as defined in [Mas-Colell \(1989\)](#). Our main result shows that whenever the measure space of agents is saturated, every non-competitive allocation can be blocked with a Walrasian objection and use this result to prove that the bargaining set, the core and the set of competitive allocations are all equivalent solution concepts.

Keywords— Walrasian objections, Bargaining set, Infinite dimensional commodity spaces, Saturation property, Lyapunov's Theorem

1 Introduction

The main purpose of these notes is to study the Walrasian objection mechanism in models of exchange economies with a measure space of agents and a separable Banach space of commodities whose positive cone has a non-empty interior.

The Walrasian objection mechanism, as defined in [Mas-Colell \(1989\)](#), is a price-based process through which the agents of an economy can block non-competitive allocations by acting solely in the pursuit of self-interest. Informally, we can describe it by imagining that every agent of the economy is asked to choose between accepting the bundle that is assigned to him through an allocation f or joining a market regulated by a price system p . While some agent may settle for f , some other may prefer to trade his initial endowments for a consumption plan that is most satisfactory at the price p . A Walrasian objection to f is reached if all the transactions relative to these plans are performable, which is to say that the total demands of the deviating agents meet their total supplies.

Our interest for Walrasian objections is motivated by the following two observations. First of all, a Walrasian objection is obtained with a totally decentralized process in which agents act in full independence. This property is particularly significant in large economies where the lack of communication and centralized institutions may put a strong restriction on the formation of coalitions and on the coordination among agents, as pointed out in [Schmeidler \(1972\)](#), [Grodal \(1972\)](#), [Vind \(1972\)](#) or [Hara \(2002\)](#). The second property we are interested in is that Walrasian objections are

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“optimal” in the sense that they cannot be counter-objected. Thus a Walrasian objection is credible in the sense that it prevents any possible reaction of other agents in the economy.

In [Mas-Colell \(1989\)](#) the author considers a non-atomic exchange economy with a finite number of commodities and proves that, under assumptions that are close to those needed for Aumann’s existence and equivalence Theorems ([1964](#); [1966](#)), every non-competitive allocation can be blocked with a Walrasian objection. This allows him to conclude that the set of competitive allocations coincides not only with the core, but also with the bargaining set of the economy, which is the collection of all the feasible allocations against which it is not possible to raise objections that are not, in turn, counter-objected. These results can be obtained thanks to Lyapunov’s Theorem that guarantees the convexity and compactness of agents’ mean demand sets.

As we move to economies with an infinite dimensional commodity space, the failure of Lyapunov’s Theorem makes it possible that some non-competitive allocations cannot be blocked with Walrasian objections. To overcome this problem, it becomes therefore necessary to strengthen the assumptions on the measure space of agents in order to guarantee a new convexifying effect on their demands with some appropriate extensions of Lyapunov’s Theorem. To this end, it bears emphasis that while other classical results can be extended to the infinite dimensional settings with some weaker convexity results, the existence of Walrasian objections necessitates the exact version of Lyapunov’s Theorem, as discussed in [Remark 3.9](#).

The importance of considering economic models with infinite dimensional commodity spaces is motivated by many relevant economic phenomena that cannot be represented in a finite dimensional setting. The date, the location or the state of nature that determine the availability of a given good, for example, are all parameters that determine agents’ choices and behaviour and that cannot be embraced all together in a model with a finite dimensional commodity space. Classical expositions on the topic with discussions on the applications and the related literature are found, for example, in [Khan and Yannelis \(1991b\)](#) and [Mas-Colell and Zame \(1991\)](#).

In these notes we give a full characterization of Walrasian objections in economies with a separable Banach space of commodities and prove that it is always possible to object non-competitive allocations with a Walrasian objection when the measure space of agents is “rich enough”. By doing this we will be able to restore the equivalence between the bargaining set and the class of competitive allocations. More precisely, this equivalence is made possible thanks to an exact extension of the Theorem of Lyapunov based on the so-called saturation property, following expedients that have been largely employed in the economic literature in [Lee \(2013\)](#), [Sun and Zhang \(2015\)](#), [Khan and Sagara \(2016\)](#) and [Jang and Lee \(2020\)](#). Since the hypothesis of saturation of the measure space is both necessary and sufficient to ensure the validity of this convexity result, as shown in [Podczeck \(2008\)](#), [Sun and Yannelis \(2008\)](#) and again in [Khan and Sagara \(2013, 2015\)](#), it seems natural to model the agents of the economy as the points of a saturated measure space.

The rest of these notes is organized as follows. After having recalled the main definitions and the notation that will be needed throughout the sections, we will describe in [2](#) a standard model of an exchange economy with a measure space of agents and a Banach space of commodities. All the assumptions we will make on the economy have been already explored in the literature and will generalize those presented in [Khan and Yannelis \(1991a\)](#). We will then adapt Mas-Colell’s definitions of objection, counter-objection and bargaining set to these new settings. In the paragraph [2.3](#) we give the definition of Walrasian objection and prove in [Theorem 2.10](#) that every Walrasian objection is justified. In [Proposition 2.11](#) we characterize the coalitions that can raise Walrasian objections at a given price while in [Theorem 2.12](#) we prove that the existence of Walrasian objections against feasible allocations depends on the existence of equilibrium prices of a properly defined correspondence.

In the Section 3, we will work under the additional assumption that the measure space of agents is saturated. This will allow us to prove that every non-competitive allocation can be blocked with a Walrasian objection in Theorem 3.6. As a corollary of this result, we will prove in Theorem 3.7 that every allocation in the bargaining set is competitive and give in Corollary 3.8 a new proof of the existence of competitive allocations in saturated economies with a separable Banach space of commodities whose positive cone has non-empty interior.

1.1 Notations and mathematical settings

We summarize here some basic definitions and notation. For any further observation and deeper mathematical result, we will primarily refer to (Khan and Yannelis, 1991b, Part I) and (Aliprantis and Border, 2006, Chapters 17, 18) as references for the general theory of integration of correspondences and Podczeck (2008), Sun and Yannelis (2008) for the specific case of integration over a saturated measure space.

Given any two topological vector spaces X and Y we write $\varphi: X \rightrightarrows Y$ to say that φ is a **correspondence** from X to Y , i.e. a map that assigns a (possibly empty) subset of Y to every $x \in X$. We say that the correspondence φ is **upper hemicontinuous** if $\{x \in X : \varphi(x) \subseteq U\}$ is open whenever U is an open subset of Y . In the specific case in which Y is a Banach space we say that φ is **weakly upper hemicontinuous** if it is upper hemicontinuous with respect to the weak topology on Y . By **graph** of a correspondence $\varphi: X \rightrightarrows Y$ we mean the set $Gr_\varphi := \{(x, y) \in X \times Y : y \in \varphi(x)\}$.

Let (T, Σ, μ) be a complete probability space. As usual, we say that a $F \in \Sigma$ is **null** (and write $F \in N(\mu)$) if $\mu(F) = 0$. When E is a Banach space, we say that a function $f: T \rightarrow E$ is **integrable** if it is Bochner integrable and write $f \in L^1(\mu, E)$. When $E = \mathbb{R}$, we also write $L^1(\mu)$ instead of $L^1(\mu, E)$.

Let $\varphi: T \rightrightarrows E$ be a correspondence with values in a Banach space. Then φ is **measurable** if $\{t \in T : \varphi(t) \cap V \neq \emptyset\}$ belongs to Σ whenever $V \subset E$ is open. When E is separable and φ is closed valued, saying that φ is measurable is equivalent with saying that its graph Gr_φ belongs to the product algebra $\Sigma \otimes \mathcal{B}(E)$, with $\mathcal{B}(E)$ being the Borelian algebra of E , see (Aliprantis and Border, 2006, Theorem 18.25). We say that $\varphi: T \rightrightarrows E$ is **integrably bounded** if there exists an integrable function $h: T \rightarrow \mathbb{R}$ such that $\sup\{\|x\| : x \in \varphi(t)\} \leq h(t)$ for almost every $t \in T$. When E is separable and φ is integrably bounded and has a measurable graph, Aumann's selection Theorem (Aliprantis and Border, 2006, Theorem 18.26) guarantees the existence of an **integrable selection** of φ , i.e. a Bochner-integrable function $f: T \rightarrow E$ such that $f(t) \in \varphi(t)$ for almost every $t \in T$. We write $\mathcal{L}_\varphi$ for the set of all integrable selections of φ and identify the functions that are almost everywhere identical. The **integral** of a correspondence $\varphi: T \rightrightarrows E$ on a set $A \in \Sigma$ is given by:

$$\int_A \varphi(t) dt := \left\{ \int_A f(t) dt : f \in \mathcal{L}_\varphi \right\}.$$

A family of correspondences $\varphi_i: T \rightrightarrows E$, $i \in \mathcal{I}$, is said to be **well dominated** if there exists an integrably bounded, weakly compact valued correspondence $K: T \rightrightarrows E$ such that $\varphi_i(t) \subseteq K(t)$ for every index $i \in \mathcal{I}$ and almost every $t \in T$.

2 The economic model

The commodity space is given by a separable Banach space E whose topological dual is E^* . A **price system** will be any $p \in E_+^* \setminus \{0\}$ with the idea that, for a $v \in E$, $p \cdot v$ gives the worth of the bundle v at the price p . We assume that there is a vector u in the interior of E_+ .

We consider an economy $\mathcal{E}$ with the following specifications.

- The agents are represented as the points of a complete probability space (T, Σ, μ) that we call the **space of agents**. Non-null elements of Σ are called **coalitions**, while μ is thought as a population measure that assigns to each $A \in \Sigma$ the proportion of all the agents in the economy that belong to A .
- Each agent t is characterized by a **consumption set** $X(t) \subseteq E_+$ and a bundle $e(t) \in X(t)$ representing his initial ownings. Integrable selections of the correspondence $X: T \rightarrow E$ will be called **consumption plans**, or **assignments**. We will write $\mathcal{L}_X$ to denote the set of all consumption plans and assume that the function $e: T \rightarrow E$ is in $\mathcal{L}_X$. A coalition S can **attain** a consumption plan f if $\int_S [e(t) - f(t)] dt \in E_+$. We will say that $f \in \mathcal{L}_X$ is **feasible**, or that f is an **allocation**, if it is attainable by the grand coalition T , i.e. if $\int_T [e(t) - f(t)] dt \in E_+$.
- For every $t \in T$ we specify a **preference relation** $\succsim_t \subseteq X(t) \times X(t)$ which we assume to be reflexive, transitive and complete. The symmetric and antisymmetric components of $\succsim_t$ are respectively called the **indifference** and **strict preference** relation and are denoted respectively by $\sim_t$ and $\succ_t$. For any $v \in X(t)$, we define agent t 's **upper contour set** $P_t(v) := \{w \in X(t) : w \succsim_t v\}$ and his **weak upper contour set** $W_t(v) := \{w \in X(t) : w \succsim_t v\}$. A **satiation point** for $\succsim_t$ is a bundle $v \in X(t)$ such that $P_t(v) = \emptyset$.

An economy will therefore be fully specified by the following tuple:

$$\mathcal{E} := [(T, \Sigma, \mu), (X(t), \succsim_t, e(t))_{t \in T}].$$

Let us take the bundle $u \in \text{int}E_+$ and restrict our attention to the prices in the simplex $\Delta := \{p \in E_+^* : p \cdot u = 1\}$. Throughout, we will always consider Δ equipped with the weak* topology, so that Δ is a convex and weakly* compact set (by Alaoglu's Theorem, see (Aliprantis and Border, 2006, Theorem 5.105)) and, by the separability of E , it is a metrizable space, see (Aliprantis and Border, 2006, Theorem 6.30). Furthermore, for every $v \in E$ one has that $p \cdot v \geq 0$ for every $p \in \Delta$ if and only if $v \in E_+$, with a strict inequality if and only if $v \in \text{int}E_+$.

For every $p \in \Delta$ and $t \in T$ we can define agent t 's **budget set** at the price p as $\beta(p, t) := \{v \in X(t) : p \cdot v \leq p \cdot e(t)\}$ and his **demand set** at p as the collections of all consumption bundles in $\beta(p, t)$ that are maximal with respect to $\succsim_t$, i.e. the set $\xi(p, t) := \{v \in \beta(p, t) : v \succsim_t \beta(p, t)\}$.¹ It follows from standard arguments that the demand set is always non-empty and weakly compact, see Lemma A.1 in the appendix or the proof of (Aliprantis and Border, 2006, Theorem 2.44).

We shall give a special importance to those allocation that can be generated in a fully decentralized economy by letting agents choose independently the consumption net-trade that they find optimal under a given budget constraint. These will be the Walrasian, or competitive, allocations.

Definition 2.1 *A **competitive equilibrium** is a pair (p, f) formed by a price system $p \in \Delta$ and a feasible allocation $f \in \mathcal{L}_X$ such that $f(t) \in \xi(p, t)$ for almost every $t \in T$. When this is the case, we say that f is **Walrasian** or **competitive** and write $f \in \mathcal{W}(\mathcal{E})$.*

More generally, we say that an assignment f is **supported** by a price system $p \in \Delta$ if for almost every $t \in T$, $p \cdot v \geq p \cdot e(t)$ whenever $v \in X(t)$ is such that $v \succsim_t f(t)$.

¹Here and throughout these notes we adopt the following convention: for any binary relation R defined on a set X , if $x \in X$ and $A \subseteq X$ then $xRA \iff xRa$ for every $a \in A$.

2.1 Further assumptions

For the rest of this paper we will assume that the economy $\mathcal{E}$ satisfies some additional properties that are listed below. Precisely, we will need to make restrictions on the consumption set correspondence $X: T \rightrightarrows E$, on the preference relations and on the set of freely disposable bundles that can be considered in our analysis.

- (A1) The correspondence $X: T \rightrightarrows E$ is integrably bounded, convex and weakly compact valued and has a measurable graph, i.e. $\{(t, v) \in T \times E : v \in X(t)\}$ belongs to $\Sigma \otimes \mathcal{B}(E)$.
- (A2) For every $t \in T$ there exists a $z_t \in X(t)$ such that $e(t) - z_t \in \text{int}E_+$.
- (A3) The correspondence $\succsim_t: T \rightrightarrows E \times E$ has a measurable graph, i.e. $\{(t, v, w) : v \succsim_t w\}$ belongs to $\Sigma \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)$.
- (A4) For every $t \in T$, $\succsim_t$ is a closed subset of $X(t) \times X(t)$ each considered with the weak topology.
- (A5) Let $t \in T$ and $v \in X(t)$. If v is a saturation point for $\succsim_t$ then $v - e(t) \in E_+$; if v is not a saturation point for $\succsim_t$ then v belongs to the weak closure of the upper-contour set $P_t(v)$.

Assumptions (A1), (A3) and (A4) are elementary continuity and measurability requirements. Assumption (A5) can be seen a weak rule of local non-satiation on non-satiation points. It also rules out the possibility that an agent facing a budget constraint may settle for a bundle whose worth is strictly smaller than his wealth. On a similar line, Assumption (A2) can be thought as a form of “strong survival” rule that ensures that $e(t)$ is not a minimal point in $X(t)$, see (Martins-da Rocha, 2003, Pg. 872).

Remark 2.2 *The Assumptions (A1) – (A5) are all natural in economic models involving a measure space of agents and an infinite dimensional commodity space. In particular, in the models considered in Khan and Yannelis (1991a), Rustichini and Yannelis (1991) and Podczeck (1997) all of the assumptions are satisfied. Still, our model is more general than the one given in Khan and Yannelis (1991a) as we do not require convexity of preferences, nor we ask that the initial endowment takes values in a norm compact subset of E . A discussion on all of these assumptions with reference to the literature can be found in Tourky and Yannelis (2001) and Martins-da Rocha (2003).*

Other economic models based on similar conditions can be found in Lee (2013), Khan and Sagara (2016, 2017) and Jang and Lee (2020).

In the following, we will rarely work directly with the demand set correspondence and prefer to make considerations on a broader correspondence that we call the extended demand. Formally, the **extended demand set** of agent t at a price p is given by $\tilde{\xi}(p, t) := \{v \in X(t) : v \succsim_t \beta(p, t)\}$, which is the set of all consumption bundles that t would prefer to any bundle in his budget set. It is clear that the standard demand set can be obtained as the intersection $\tilde{\xi}(p, t) \cap \beta(p, t)$.

The following lemma, whose proof can be found in Appendix A.2, can be obtained using some standard arguments.

Lemma 2.3 *For every $t \in T$, $p \in \Delta$ and $v \in X(t)$ the following conditions are equivalent:*

1. $v \in \tilde{\xi}(p, t)$;
2. for every $w \in X(t)$, $w \succsim_t v \Rightarrow p \cdot w \geq p \cdot e(t)$;
3. for every $w \in X(t)$, $w \succ_t v \Rightarrow p \cdot w > p \cdot e(t)$.

As a consequence of Lemma 2.3, saying that $f \in \mathcal{L}_X$ is supported by a price system p is equivalent with saying that $f(t) \in \tilde{\xi}(p, t)$ for almost every $t \in T$. In particular, this means that an allocation is Walrasian only if it is supported by a price system $p \in \Delta$.

2.2 Objections, counter-objections and bargaining set

Let us fix an assignment $f \in \mathcal{L}_X$. Intuitively, a group of agents will object f if they find a better way to allocate among themselves the goods that they are initially endowed with. As a consequence, these agents would prefer leave the economy and form an independent market rather than accepting the redistribution of resources described by f . This intuition is formalized as follows.

Definition 2.4 *A coalition S can **object**, or **block**, f if there exists a $g \in \mathcal{L}_X$ satisfying the following conditions.*

- (i) S can attain g , i.e. $\int_S [e(t) - g(t)] dt \in E_+$;
- (ii) $g(t) \succ_S f(t)$ for almost every $t \in S$;
- (iii) $\mu(\{t \in S : g(t) \succ_t f(t)\}) > 0$.

In this case we say that (S, g) is an **objection** to f and write $(S, g) \in Ob(f)$.

The **core** of the economy is the set $\mathcal{C}(\mathcal{E})$ formed by all the allocations that cannot be objected.

Any objection (S, g) to f induces a new consumption plan $\hat{f}_{S,g}$ that assigns to each agent $t \in T$ the vector:

$$\hat{f}_{S,g}(t) := \begin{cases} g(t), & \text{if } t \in S; \\ f(t), & \text{otherwise.} \end{cases}$$

In what follows we will refer to $\hat{f}_{S,g}$ as the **assignment induced by the objection** (S, g) . It shall be stressed that $\hat{f}_{S,g}$ may not be feasible even when f is so.

Intuitively, $\hat{f}_{S,g}$ represents the virtual² consumption plan that emerges after the agents in S have decided to deviate from f and re-allocate their resources according to g . As such, the objection $(S, g) \in Ob(f)$ will not be credible, nor stable, if a significant share of agents can in turn improve upon $\hat{f}_{S,g}$. We formalize this intuition by saying that a coalition Q can **counter-object** (S, g) if it can block the induced allocation $\hat{f}_{S,g}$ via some $h \in \mathcal{L}_X$. In other words, we have given the following definition.

Definition 2.5 *Let $(S, g) \in Ob(f)$. A coalition Q **counter-objects** (S, g) if and only if there is a $h \in \mathcal{L}_X$ satisfying the following conditions:*

- (i) h is attainable by Q , i.e. $\int_Q [e(t) - h(t)] dt \in E_+$;
- (ii) for every $t \in Q$, $h(t) \succ_t g(t)$ when $t \in S$ and $h(t) \succ_t f(t)$ when $t \notin S$;
- (iii) $\mu(\{t \in Q : h(t) \succ_t g(t)\}) > 0$ or $\mu(\{t \in Q : h(t) \succ_t f(t)\}) > 0$.

In this case we call (Q, h) a **counter-objection** to (S, g) and write $(Q, h) \in Cob(S, g; f)$.

We observe in particular that, by the transitivity of preference relations, if $(S, g) \in Ob(f)$ and (Q, h) is a counter-objection to (S, g) then (Q, h) is also an objection to f .

²The adjective *virtual* is borrowed from the register used in physics where it indicates a potential displacement that may not meet the constraint of the model. In this case it is used to stress the idea that the consumption plan $\hat{f}_{S,g}$ may not be performable.

Definition 2.6 An objection (S, g) to a consumption plan f is **justified** if there is no counter-objection to it. The **bargaining set** of the economy $\mathcal{E}$ is the set $BS(\mathcal{E})$ formed by all the allocations that have no justified objections.

In other words, the allocation f belongs to the bargaining set if whenever (S, g) is an objection to f there exists a counter-objection against it. Clearly, being each allocation in the core impossible to object in the first place, the bargaining set will contain every core allocation. This observation is formalized and extended in the following proposition, whose proof is given in A.3.

Proposition 2.7 If $f \in \mathcal{L}_X$ is supported by a $p \in \Delta$ then $Ob(f) = \emptyset$.

In particular the following inclusions hold:

$$\mathcal{W}(\mathcal{E}) \subseteq \mathcal{C}(\mathcal{E}) \subseteq BS(\mathcal{E}).$$

Remark 2.8 The definition of counter-objection we have given differs slightly from the one presented in Mas-Colell (1989). Formally, if $(S, g) \in Ob(f)$, we allow a coalition Q to counter-object (S, g) via an assignment h even when a significant share of agent in Q is indifferent between h and g . This was not the case in Mas-Colell (1989) where the bullet (ii) in the Definition 2.5 is replaced by the stronger requirement:

$$t \in Q \cap S \Rightarrow h(t) \succ_t g(t), \quad \text{and } t \in Q \setminus S \Rightarrow h(t) \succ_t f(t).$$

In Mas-Colell's framework, however, using the monotonicity and continuity of preference relations one can prove that an objection (S, g) is counter-objected in the sense of Mas-Colell if and only if it is counter-objected in the sense of Definition 2.5. As a consequence, the set of justified objections remains the same under any of the two definitions adopted. Further observations and remarks on these two different notions of counter-objections are found in Vohra (1991) and Yamazaki (1995).

2.3 Walrasian objections

As mentioned in the introduction, the idea of Walrasian objections is that of imposing a price system p and letting each individual decide whether he wants to keep the bundle assigned to him by an allocation f or to trade his initial resources in a market regulated by the price system p . For this to be possible, we expect each agent t to choose between consuming $f(t)$ or some bundle in his demand set $\xi(p, t)$ that he prefers to $f(t)$. This intuition is formalized as follows.

Definition 2.9 An objection (S, g) to f is **Walrasian**, or **competitive**, if there exists a price system p such that, for almost every $t \in T$ and every $v \in X(t)$, one has:

- $p \cdot v \geq p \cdot e(t)$ whenever $t \in S$ and $v \succ_t g(t)$;
- $p \cdot v \geq p \cdot e(t)$ whenever $t \notin S$ and $v \succ_t f(t)$.

With the definitions given above and the results in Lemma 2.3 it is clear that an objection $(S, g) \in Ob(f)$ is Walrasian if and only if $g(t) \in \tilde{\xi}(p, t)$ for almost every $t \in S$ and $f(t) \in \tilde{\xi}(p, t)$ for almost every $t \notin S$. Equivalently, we can say that (S, g) is Walrasian if the allocation $\hat{f}_{S, g}$ it induces is supported by a price system p . On this line, the following Theorem can be proved thanks to Lemma 2.3. The proof is found in A.4.

Theorem 2.10 Every Walrasian objection is justified.

In the light of Theorem 2.10 we can say that a feasible allocation f will not be in the bargaining set whenever there exists a Walrasian objection against it.

In order to make some further observation let us introduce now some additional notation. For any $f \in \mathcal{L}_X$ and $p \in \Delta$ let us define the sets:

$$C_f(p) := \{t \in T : \xi(p, t) \succ_t f(t)\}, \quad D_f(p) := \{t \in T : \xi(p, t) \succsim_t f(t)\}.$$

Intuitively, an agent t will belong to $C_f(p)$ (resp. $D_f(p)$) if he strictly (resp. weakly) prefers what she obtains by trading $e(t)$ at the price p over the bundle $f(t)$. Please observe that a feasible allocation f will be Walrasian if and only if there is a price system p such that $\mu(C_f(p)) = 0$.

Using these definitions we can give a first characterization of Walrasian objections whose proof is given in A.5.

Proposition 2.11 *Let f be an allocation that is not competitive. Then a pair (S, g) is a Walrasian objection to f if and only if there exists a $p \in \Delta$ such that:*

1. $g(t) \in \xi(p, t)$ for almost every $t \in S$;
2. $C_f(p) \subseteq S \subseteq D_f(p)$ almost everywhere³;
3. $\int_S [e(t) - g(t)] dt = 0$.

With the proposition above it becomes possible to characterize the price systems that will be significant in the Walrasian objection process. Precisely, given a non-competitive allocation f we can define a proper correspondence $\Psi_f: \Delta \rightarrow E$ and show that the prices that can generate a Walrasian objection to f are those obtained as equilibrium prices for Ψ_f , i.e. those $p \in \Delta$ such that $\Psi_f(p) \cap E_+ \neq \emptyset$. This is done in the following theorem.

Theorem 2.12 *Let f be an allocation that is not competitive. Then there is a Walrasian objection to f at a price system p if and only if E_+ intersects the set:*

$$\Psi_f(p) := \left\{ \int_S [e(t) - g(t)] dt : C_f(p) \subseteq S \subseteq D_f(p), g \in \mathcal{L}_{\xi(p, \cdot)} \right\}.$$

To get a better intuition of Theorem 2.12 let us go back to the naive description of the Walrasian objection mechanism and try to give a formal description of individuals' actions. When agents face an allocation f and a price system p , any $t \in C_f(p)$ will select a bundle in his demand set, every $t \notin D_f(p)$ will remain inactive and each $t \in D_f(p) \setminus C_f(p)$ can either select a bundle in $\xi(p, t)$ or remain inactive. The net-trades that follow can be described by means of the **modified excess of supply** correspondence $\psi_f: \Delta \times T \rightarrow E$ that assigns to each $p \in \Delta$ and $t \in T$ the set:

$$\psi_f(p, t) := \begin{cases} \{e(t)\} - \xi(p, t), & \text{if } t \in C_f(p); \\ (\{e(t)\} - \xi(p, t)) \cup \{0\}, & \text{if } t \in D_f(p) \setminus C_f(p); \\ \{0\}, & \text{otherwise.} \end{cases}$$

It should be easy to check that the integral $\int \psi_f(p, t) dt$ coincides with the set $\Psi_f(p)$ described in Theorem 2.12. We can therefore conclude that there exists a Walrasian objection to f at a price system p if and only if there exists a disposable bundle in the average modified excess of supply at p , i.e. if $\int \psi(p, t) dt \cap E_+ \neq \emptyset$.

³An inclusion $A \subseteq B$ holds almost everywhere if $\mu(B \setminus A) = 0$.

3 Existence of Walrasian objections in saturated economies

As it has been shown in Theorem 2.12, the problem of the existence of a Walrasian objection to an allocation f can be reduced to that of the existence of equilibrium prices for the correspondence Ψ_f , i.e. those $p \in \Delta$ such that $\Psi_f(p) \cap E_+ \neq \emptyset$. Therefore, a natural way to approach this problem is to look for conditions under which Ψ_f satisfied some infinite dimensional extension of the Gale-Debreu-Nikaidô Lemma.

In this section we try to prove the existence of Walrasian objections by considering the following version of the Lemma whose proof can be found in Yannelis (1985), see also Mehta and Tarafdar (1987) and (Florenzano, 2003, Chapter 2).

Theorem 3.1 *Let $\varphi: \Delta \rightarrow E$ be a correspondence that satisfies the following assumptions.*

1. $\varphi(p)$ is non-empty, convex and weakly compact for every $p \in \Delta$;
2. φ is weakly upper hemicontinuous;
3. for all $p \in \Delta$ there exists a $v \in \varphi(p)$ such that $p \cdot v \geq 0$.

Then there exists a $\bar{p} \in \Delta$ and a $\bar{v} \in \varphi(\bar{p})$ such that $q \cdot \bar{v} \geq 0$ for every $q \in \Delta$.

Since $E_+ = \{v \in E : q \cdot v \geq 0 \ \forall q \in \Delta\}$, if we showed that for some allocation f the correspondence Ψ_f meets all the requirements above, we could combine the Theorems 3.1 and 2.12 to prove the existence of a Walrasian objection to f . As we will see, however, this is not an easy task as two main difficulties arise: first, we have to find conditions under which $\Psi_f(p)$ is convex and weakly compact for every $p \in \Delta$; second we will need to short-circuit the possible lack of upper hemicontinuity of Ψ_f that is typical of the infinite dimensional settings. We will address these two points in the next paragraphs following ideas that have already been successfully applied to other similar problems in Podczeck (1997) and Lee (2013).

3.1 The saturation property

In the finite dimensional case, as it is in Mas-Colell (1989), the convexity and compactness of $\Psi_f(p)$ is guaranteed whenever (T, Σ, μ) is non-atomic. This is because, as we mentioned at the end of the paragraph 2.3, the set $\Psi_f(p)$ can be described as the integral $\int \psi_f(p, t) dt$ and so its compactness and convexity follow from the Theorem of Lyapunov, see (Aliprantis and Border, 2006, Theorem 13.13). When the commodity space is infinite dimensional, however, the failure of Lyapunov's Theorem imposes that some stronger assumptions on (T, Σ, μ) have to be made in order to obtain a similar convexifying effect. In this paragraph, we study the specific case in which the space of agents is **saturated**, a property that has been introduced in Hoover and Keisler (1984) and that is now central in the economic literature. For a broad description of this property and its applications we mention the survey Fajardo and Keisler (2004); Keisler and Sun (2009) and its references, similar and more general conditions are studied in Greinecker and Podczeck (2013), Khan and Sagara (2013) and Urbinati (2019).

Let us focus on the space (T, Σ, μ) . For any $A \in \Sigma$ we can define the measure $\mu_A: \Sigma \rightarrow \mathbb{R}$ that assigns $\mu(A \cap B)$ to any $B \in \Sigma$. By doing so, we can identify the space $L^1(\mu_A)$ with that of the integrable functions whose support is contained in A , i.e. those $f \in L^1(\mu)$ such that $\mu(\{t \in T \setminus A : f(t) \neq 0\}) = 0$. The notion of a saturated measure space depends on the density of these $L^1(\mu_A)$, i.e. on the smallest among the cardinalities of all dense subsets of $L^1(\mu_A)$.

Definition 3.2 A measure space (T, Σ, μ) is **saturated** if for every $A \in \Sigma \setminus N(\mu)$ the space $L^1(\mu_A)$ is non-separable (it does not have any countable subset).

This definition generalizes that of non-atomicity of a measure space in the sense that every saturated measure space is automatically non-atomic. Furthermore, for a saturated measure space the following result provides us the tools that in the finite dimensional case were obtained from the Theorem of Lyapunov. For a proof and further discussions see (Sun and Yannelis, 2008, Proposition 1.6) or (Podczeck, 2008, Fact at page 839).

Theorem 3.3 Let (T, Σ, μ) be a saturated measure space and $\varphi: \Delta \times T \rightrightarrows E$ a weakly compact valued correspondence for which all the following conditions hold.

1. $\varphi(p, \cdot): T \rightrightarrows E$ is measurable for every $p \in \Delta$;
2. $\varphi(\cdot, t): \Delta \rightrightarrows E$ is weakly upperhemicontinuous for almost every $t \in T$;
3. $(\varphi(p, \cdot))_{p \in \Delta}$ is a well dominated⁴ family of correspondences.

Then the correspondence $\int \varphi(\cdot, t) dt: \Delta \rightrightarrows E$ is weakly upper hemicontinuous and it is convex and weakly compact valued.

In the light of this theorem, by assuming that (T, Σ, μ) is a saturated measure space we can reduce the study of the correspondence $\Psi_f: \Delta \rightrightarrows E$ to that of $\psi_f: \Delta \times T \rightrightarrows E$.

3.2 The modified extended demand

Unfortunately, for almost every $t \in T$ the standard demand correspondence $\xi(\cdot, t): \Delta \rightrightarrows E$ will usually fail to be weakly upper hemicontinuous. This means that for any non-competitive allocation f neither the modified excess of demand ψ_f will be weakly upper hemicontinuous and so Theorem 3.3 cannot be directly applied to ψ_f .

In order to overcome this difficulties, in this paragraph we move our attention from ψ_f to a new correspondence $\tilde{\psi}_f: \Delta \times T \rightrightarrows E$ that assigns to each $p \in \Delta$ and $t \in T$ the set:

$$\tilde{\psi}_f(p, t) := \begin{cases} \{e(t)\} - \tilde{\xi}(p, t), & \text{if } t \in C_f(p); \\ (\{e(t)\} - \tilde{\xi}(p, t)) \cup \{0\}, & \text{if } t \in D_f(p) \setminus C_f(p); \\ \{0\}, & \text{otherwise.} \end{cases}$$

Please observe that $\tilde{\psi}_f$ dominates ψ_f in the sense that $\psi_f(p, t) \subseteq \tilde{\psi}_f(p, t)$ for almost every $t \in T$ and $p \in \Delta$. Furthermore, differently from ψ_f , for almost every $t \in T$ the correspondence $\tilde{\psi}_f(\cdot, t): \Delta \rightrightarrows E$ is weakly upper hemicontinuous (see Lemma A.6 for a proof) and so it will be easier to use it in fixed-point arguments.

The following proposition, whose proof is given in A.7, is a simple reformulation of Theorem 2.12.

Proposition 3.4 Let f be an allocation that is not competitive. Then there is a Walrasian objection to f at a price system p if and only if E_+ intersects the set:

$$\tilde{\Psi}_f(p) := \int \tilde{\psi}(p, t) dt.$$

⁴Recall that the family of correspondence $\varphi(p, \cdot): T \rightrightarrows E$ is well dominated if there exists an integrably bounded, weakly compact-valued $K: T \rightrightarrows E$ such that $\varphi(p, t) \subseteq K(t)$ for almost every $t \in T$.

Thanks to Proposition 3.4 we can base our analysis of Walrasian objections on the more regular correspondence $\tilde{\psi}_f$.

Remark 3.5 *The lack of continuity of the demand correspondence represents one of the main difficulties in economic models with a measure space of agents and an infinite dimensional commodity space. From a technical point of view, it has to do with the fact that for an infinite dimensional space E the evaluation map $(p, v) \mapsto p \cdot v$, for $p \in E^*$ and $v \in E$, is no longer jointly continuous when we endow E^* and E respectively with the weak* and the weak topology, see (Aliprantis and Border, 2006, pp.241–242).*

It is to overcome these difficulties that in Khan and Yannelis (1991a) the authors restrict their attention to a norm compact subset of the commodity space. Here, we have decided to follow an approach inspired by Podczeck (1997) which allows a greater level of generality.

3.3 The main Theorem

With all the results and observations made we can now move to the main result of this section. The main idea is to use Theorem 3.3 to prove that the correspondence $\tilde{\Psi}_f: \Delta \rightarrow E$ meets all the requirements of Theorem 3.1. This way, it will be possible to apply Proposition 3.4 to obtain the following theorem.

Theorem 3.6 *Suppose that (T, Σ, μ) is a saturated measure space and that f is a feasible allocation that is not competitive. Then there exists a Walrasian objection to f .*

Once the effectiveness of the Walrasian objection mechanism has been established, many significant results can be obtained from Theorem 3.6. As a way of illustration, we can combine Proposition 2.7 and Theorem 3.6 to prove the following equivalence, see A.9 for a proof.

Theorem 3.7 *Suppose that (T, Σ, μ) is a saturated measure space. Then:*

$$\mathcal{W}(\mathcal{E}) = \mathcal{C}(\mathcal{E}) = BS(\mathcal{E}).$$

Following the observations in Mas-Colell (1989), one can also obtain as a corollary of Theorem 3.6 the existence of competitive equilibria in saturated economies with a separable Banach space of commodities having an interior point of the positive cone. The proof of the following Theorem is given in A.10.

Theorem 3.8 *Suppose that (T, Σ, μ) is a saturated measure space. Then there exists a competitive allocation.*

The result above is not new to the literature and has already been obtained with different techniques and more general assumptions in Lee (2013), Khan and Sagara (2017) and Jang and Lee (2020) as extensions of those in Rustichini and Yannelis (1991). It bears emphasis, however, that all of the proofs in the mentioned articles rely heavily on the non-emptiness of the interior of the positive cone, as it is in our settings.

Remark 3.9 *As a corollary of Theorem 3.6 we have obtained both the core-Walras equivalence (Theorem 3.7) and the existence of competitive allocations (Theorem 3.8). In the finite dimensional case, both of these results are proved thanks to Lyapunov's Theorem respectively in Aumann (1964, 1966). It shall be pointed out, however, that in the infinite dimensional settings the two Theorems can be obtained without imposing the saturation of the measure space of agents.*

For the existence of competitive allocations, for example, it is sufficient that the integral of the demand set $\int \xi(p, t) dt$ is convex and compact for every $p \in \Delta$ and $t \in T$. This can be obtained by assuming that each agent has convex preferences, as in [Khan and Yannelis \(1991a\)](#), [Bewley \(1991\)](#) and [Noguchi \(1997\)](#). Other techniques are developed in [Podczeck \(1997\)](#) and [Martins-da Rocha \(2003\)](#).

Even the core-Walras equivalence can be obtained with a non-saturated measure space of agents. As pointed out in [Armstrong and Richter \(1984\)](#), in fact, for this result the Theorem of Lyapunov is needed in a weaker form that can more easily be extended to infinite-dimensional case. As examples in which this idea is pursued in economies with many commodities we mention [Cheng \(1991\)](#), [Martellotti \(2008\)](#) and [Donnini and Graziano \(2009\)](#).

We stress that the results of Theorem 3.6 cannot be proved with the same expedients mentioned above. It is well known, in fact, that the existence of Walrasian objections depends on deeper mathematical results even in the case of a finite number of commodities. This is well explained in [Anderson et al. \(1997a\)](#) and [Grodal \(2009\)](#). To grasp the intuition behind this, it is enough to go back to Theorem 2.12 and observe that if (T, Σ, μ) is non-saturated, there is no way to guarantee the convexity of the sets $\Psi_f(p)$ just by assuming the convexity of preferences.

4 Concluding remarks

We have studied the Walrasian objection mechanism and Mas-Colell's bargaining set in economic models with a measure space of agents and an infinite dimensional commodity space. Each one of these two topics, that are intrinsically related, is of interest per se and deserves independent observations and remarks.

The Walrasian objection mechanism represents a valuable tool to study the formation of coalitions in a fully decentralized, large economy. In the absence of central institutions, in fact, it could be impossible for a large number of individuals to reach a private agreement and improve upon a given allocation. A Walrasian objections, however, does not require any form of communication among agents and can be seen as a spontaneous reaction to a given price system in which every individual chooses autonomously whether or not he wants to join the deviating coalition. Objections of this kind, i.e. those in which agents observe a public announcement and choose autonomously whether or not they intend to join the deviating coalition, are called "anonymous" in [Hara \(2002\)](#) as they ensure that all agents with the same characteristics are treated the same way. In our framework, we may adapt Hara's definition by saying that an objection (S, g) to an allocation f is **anonymous** if: (a) for almost every $t, s \in S$, $g(t) \succsim_t e(t) + (g(s) - e(s))$ whenever $e(t) + (g(s) - e(s)) \in X(t)$, (b) for almost every $t \in T \setminus S$ and $s \in S$, $f(t) \succsim_t e(t) + (g(s) - e(s))$ whenever $e(t) + (g(s) - e(s)) \in X(t)$. As a corollary of Theorem 3.6 we can prove that in a saturated economy all non-competitive allocations with the equal treatment property are blocked by an anonymous objection. It would be of interest, however, to study anonymous objections in an infinite dimensional settings where the equivalence $\mathcal{W}(\mathcal{E}) = BS(\mathcal{E})$ does not hold.

With a different approach, one could try to bypass the lack of communication in large economies by considering only objections raised by coalitions that are limited in size. Pursuing this idea, [Schmeidler \(1972\)](#), [Grodal \(1972\)](#) and [Vind \(1972\)](#) prove that in non-atomic economies with a finite number of commodities all non-competitive allocations can be blocked by coalitions that are arbitrarily small or arbitrarily large in size. Even in this case, it is possible to reproduce the same results in an infinite dimensional setting under the additional assumption that the measure space of agents is saturated. This is done in [Khan and Sagara \(2013\)](#), [Greinecker and Podczeck \(2013\)](#) and [Urbinati \(2019\)](#). It shall be stressed, however, that the veto mechanism that follows from limiting the size of objecting

coalitions is too weak to be used in a two-step process as the one that defines the bargaining set. Even in the case of finite dimensional economies, in fact, the examples in [Schjødtt and Sloth \(1994\)](#) prove that it is not possible to bound the size of competitive, and hence justified, objections. A deeper analysis of these issues is found in [Hervés-Estévez and Moreno-García \(2015\)](#).

The notion of bargaining set can be seen as way to weaken the veto mechanism that defines the core by considering only objections that are not counter-objected. Different notions of bargaining set could be obtained by imposing specific conditions on the class of objections and counter-objections that are considered from time to time. As a way of illustration, we observe that the equivalence $BS(\mathcal{E}) = \mathcal{W}(\mathcal{E})$ proved in [Theorem 3.8](#) relies on the fact that in our definition of objection it is possible that some agent is indifferent between the objected allocation and the new one. If we considered only objections in which all agents have strict preferences we would leave the core unaltered but we would define a bargaining set that is strictly larger than $BS(\mathcal{E})$.

As observed in ([Mas-Colell, 1989](#), Remark 2), the equivalence between the bargaining set and Walrasian allocations can be re-established by adapting the idea of a “leader” a’ la Aumann-Maschler from [Geanakoplos \(1978\)](#). Formally, for any $\varepsilon > 0$ we can say that (S, g) is an ε -objection to an allocation f with leader K if $g(t) \succ_t f(t)$ for every $t \in S$ and K is a subset of S with measure $\varepsilon\mu(S)$. A feasible allocation f belongs to the ε -bargaining set $\varepsilon - BS(\mathcal{E})$ if for every ε -objection to f with leader K there exists a counter-objection (Q, h) such that $\mu(Q \cap K) > 0$. Following the same arguments as in ([Mas-Colell, 1989](#), Remark 2), one shows that if the measure space of agents is non-atomic and (S, g) is a justified objection to f then f does not belong to $\varepsilon - BS(\mathcal{E})$. [Theorem 3.6](#) can then be used to prove that in a saturated measure space with a Banach space of commodities the equivalence $\varepsilon - BS(\mathcal{E}) = \mathcal{W}(\mathcal{E})$ holds for every $\varepsilon > 0$.

Last, we stress that the equivalence $BS(\mathcal{E}) = \mathcal{W}(\mathcal{E})$ obtained in [3.7](#) cannot be obtained as the limit of a series of replicated economies even in the finite dimensional settings. This is shown in [Anderson et al. \(1997b\)](#) and again in [Hervés-Estévez and Moreno-García \(2018\)](#). In the light of this, the choice to work with saturated measure spaces of agents, which cannot approximated with series of finite sets of agents, does not seem to be a strict imposition.

A Appendix

A.1 Proofs of the results in Section 2

Lemma A.1 *For every $t \in T$ and $p \in \Delta$ the set $\xi(p, t)$ is non-empty and weakly compact.*

Proof. The budget set $\beta(p, t)$ is a non-empty, weakly closed subset of $X(t)$. Since $X(t)$ is weakly compact by Assumption (A1), $\beta(p, t)$ is weakly compact too.

For every $v \in \beta(p, t)$, let us define $K(v)$ as $W_t(v) \cap \beta(p, t)$, which is weakly closed (and hence weakly compact) by the Assumption (A4), and observe that $\xi(p, t)$ can be written as the intersection $\bigcap \{K(v) : v \in \beta(p, t)\}$. Furthermore, for every finite $F \subseteq \beta(p, t)$ we can choose a $v_F \in F$ such that $v_F \succsim_t v$ for every other $v \in F$, which means that $v_F \in \bigcap_{v \in F} K(v)$. But then $(K(v))_{v \in \beta(p, t)}$ is a family of weakly compact sets with the finite intersection property and so $\bigcap \{K(v) : v \in \beta(p, t)\}$ is non-empty and weakly compact as claimed. ■

Lemma A.2 (2.3) *For every $t \in T$, $p \in \Delta$ and $v \in X(t)$ the following conditions are equivalent:*

1. $v \in \tilde{\xi}(p, t)$;
2. for every $w \in X(t)$, $w \succsim_t v \Rightarrow p \cdot w \geq p \cdot e(t)$;
3. for every $w \in X(t)$, $w \succ_t v \Rightarrow p \cdot w > p \cdot e(t)$.

Proof.

(1 $\Rightarrow$ 2): Let $w \in X(t)$ be such that $p \cdot w < p \cdot e(t)$ and $w \succsim_t v$. Since $p \cdot (w - e(t)) < 0$ implies that $w - e(t) \notin E_+$, we know that w is not a satiation point for $\succsim_t$ and that it belongs to the weak closure of $P_t(w)$ (Assumption (A5)). This means that we can choose a $w' \in X(t)$ such that $p \cdot w' < p \cdot e(t)$ and $w' \succ_t w$. But then w' is a bundle in $\beta(p, t)$ that is strictly preferred to v , which means that $v \notin \tilde{\xi}(p, t)$ and that (1) is violated.

(2 $\Rightarrow$ 3): Let $w \in X(t)$ be such that $w \succ_t v$ and $p \cdot w \leq p \cdot e(t)$. If we take $z_t \in X(t)$ as in Assumption (A2) and define the vector $w_\theta := \theta w + (1 - \theta)z_t$ for every $\theta \in (0, 1)$, we will always have that $p \cdot w_\theta < p \cdot e(t)$. But then, by Assumption (A4), for a θ sufficiently close to 1 we will have that w_θ is such that $w_\theta \succ_t v$ even though $p \cdot w_\theta < p \cdot e(t)$. This violates point (2).

(3 $\Rightarrow$ 1): Suppose that (3) holds and let $w \in \beta(p, t)$. Since it cannot be that $w \succ_t v$ (by (3)), it must be that $v \succsim_t w$. By the generality of w we conclude that $v \in \tilde{\xi}(p, t)$. ■

Proposition A.3 (2.7) *If $f \in \mathcal{L}_X$ is supported by a $p \in \Delta$ then $Ob(f) = \emptyset$.*

In particular the following inclusions hold:

$$\mathcal{W}(\mathcal{E}) \subseteq \mathcal{C}(\mathcal{E}) \subseteq BS(\mathcal{E}).$$

Proof. Let $f \in \mathcal{L}_X$ be supported by a $p \in \Delta$ and assume that $S \in \Sigma$ and $g \in \mathcal{L}_X$ are such that $g(t) \succsim_t f(t)$ for almost every $t \in S$ and that $S' := \{t \in S : g(t) \succ_t f(t)\}$ is non-null. We claim that $\int_S [e(t) - g(t)] dt \notin E_+$, so that (S, g) cannot be an objection to f . To prove this, observe that $p \cdot [e(t) - g(t)] \leq 0$ for almost every $t \in S$, with a strict inequality if $t \in S'$. But then $p \cdot \int_S [e(t) - g(t)] dt = \int_S p \cdot [e(t) - g(t)] dt < 0$, which means that $\int_S [e(t) - g(t)] dt \notin E_+$ as claimed.

To prove the inclusions let $f \in \mathcal{L}_X$ be feasible. If $f \in \mathcal{W}(\mathcal{E})$ then it is supported by a price system $p \in \Delta$ and, by the argument above, $Ob(f)$ is empty. This means that $f \in \mathcal{C}(\mathcal{E})$. At the same time,

if $f \in \mathcal{C}(\mathcal{E})$ then there are no objections to f and, in particular, no justified objections. This gives $\mathcal{C}(\mathcal{E}) \subseteq BS(\mathcal{E})$. $\blacksquare$

Theorem A.4 (2.10) *Every Walrasian objection is justified.*

Proof. Let (S, g) be a Walrasian objection to a $f \in \mathcal{L}_X$ and let $\hat{f}_{S,g}$ be the assignment it induces. By construction, (S, g) is Walrasian at a price $p \in \Delta$ if and only if $\hat{f}_{S,g}$ is supported by p . But then, by Proposition 2.7, there are no objections to $\hat{f}_{S,g}$, which means that $Cob_f(S, g) = \emptyset$ and so that (S, g) is justified as claimed. $\blacksquare$

Proposition A.5 (2.11) *Let f be an allocation that is not competitive. Then a pair (S, g) is a Walrasian objection to f if and only if there exists a $p \in \Delta$ such that:*

1. $g(t) \in \xi(p, t)$ for almost every $t \in S$;
2. $C_f(p) \subseteq S \subseteq D_f(p)$ almost everywhere⁵;
3. $\int_S [e(t) - g(t)] dt = 0$.

Proof. Let (S, g) be a Walrasian objection to f and $p \in \Delta$ the relative price system. By Lemma 2.3, for almost every $t \in S$ have that $g(t) \in \tilde{\xi}(p, t)$ and that $p \cdot g(t) \geq p \cdot e(t)$. At the same time we know that S can attain g , which is to say that $\int_S [e(t) - g(t)] dt \in E_+$. It follows that $\int_S p \cdot [e(t) - g(t)] dt = 0$ and so $p \cdot g(t) = p \cdot e(t)$ for almost every $t \in S$. This proves that $g(t) \in \xi(p, t)$ for almost every $t \in S$, as claimed in point (1). To prove point (2), recall that $g(t) \succsim_t f(t)$ for almost every $t \in S$ and so, by the argument above, $\xi(p, t) \succsim_t f(t)$. This proves that $S \subseteq D_f(p)$ almost everywhere. At the same time, for almost every $t \notin S$ we know that $v \succsim_t f(t) \Rightarrow v \notin \xi(p, t)$ by Lemma 2.3 and so it cannot be that $\xi(p, t) \succsim_t f(t)$. It follows that $C_f(p) \subseteq S$ almost everywhere. Last, point (3) follows directly from the definition of objection.

Suppose now that for (S, g) and p all the three points are satisfied. We first need to prove that $(S, g) \in Ob(f)$. Point (3) guarantees that S can attain g . Furthermore, for almost every $t \in S$ we know that $g(t) \in \xi(p, t)$ (point (1)) and that $\xi(p, t) \succsim_t f(t)$ (point (2)). Last we observe that $C_f(p)$ is non-null (otherwise f would be competitive at the price system p) and that $g(t) \succsim_t f(t)$ for almost every $t \in C_f(p)$. This concludes that proof that $(S, g) \in Ob(f)$. To prove that (S, g) is Walrasian at the price p it is enough to observe that by construction the assignment $\hat{f}_{S,g}$ induced by (S, g) is supported by p . $\blacksquare$

A.2 Proofs of the results in Section 3

Lemma A.6 *Let f be a non-competitive allocation. Then the following conditions hold.*

1. For every $p \in \Delta$ the correspondence $\tilde{\psi}_f(p, \cdot): T \rightarrow E$ has a measurable graph.
2. For almost every $t \in T$ the correspondence $\tilde{\psi}_f(\cdot, t): \Delta \rightarrow E$ is weakly upper hemicontinuous.
3. For almost every $p \in \Delta$ there exists a $g_p \in \mathcal{L}_X$ such that $g_p(t) \in \psi_f(p, t)$ for almost every $t \in T$.

Proof. To prove point (1) let us fix a $p \in \Delta$. The measurability of the graphs of the budget set correspondence and of the demand set correspondence have been proved in (Khan and Yannelis, 1991a, pg.243). Since $\tilde{\xi}(p, t) = [X(t) \setminus \beta(p, t)] \cup \xi(p, t)$ for every $t \in T$, the correspondence $\tilde{\xi}(p, \cdot)$ has a

⁵An inclusion $A \subseteq B$ holds almost everywhere if $\mu(B \setminus A) = 0$.

measurable graph too. To conclude, it is enough to observe that the graph of $\tilde{\psi}_f(p, \cdot)$ can be written as the union of the two following measurable sets.

$$\{(t, v) \in T \times E : t \in D_f(p), v \in \tilde{\xi}(p, t)\}, \quad \{(t, 0) \in T \times E : t \notin C_f(p)\}.$$

To prove point (2) let us fix a $t \in T$. We first prove that the extended demand $\tilde{\xi}(\cdot, p)$ is weakly upper hemicontinuous. The first thing we observe is that for every $p \in \Delta$, $\tilde{\xi}(p, t)$ is a weakly closed, and hence weakly compact, subset of $X(t)$. To prove the upper hemicontinuity of the correspondence $\tilde{\xi}(\cdot, \cdot)$ it will then be enough to show that it has a closed graph and apply (Aliprantis and Border, 2006, Theorem 17.16). Let $(p_n, v_n) \in Gr_{\tilde{\xi}(\cdot, t)}$ be such that $p_n \rightarrow p_0$ and $v_n \rightarrow v_0$ for some $p_0 \in \Delta$ and $v_0 \in E$ and assume by contradiction that $v_0 \notin \tilde{\xi}(p_0, t)$. By Lemma 2.3, this means that there exists a $w \in X(t)$ such that $w \succ_t v$ and $p_0 \cdot w < p_0 \cdot e(t)$. Since $p_n \rightarrow p_0$, for a sufficiently large n we will have that $p_m \cdot w < p_m \cdot e(t)$ for every $m > n$. Another application of Lemma 2.3 will then give us that $v_m \succ_t w$ for all $m > n$. Call $K_w := \{v \in X(t) : w \succ_t v\}$ and observe that $v_0 \in K_w$ while $v_n \notin K_w$ for every $n \in \mathbb{N}$. But this is a contradiction with the fact that K_w is weakly open and v_0 is the weak limit of the v_n 's.

To prove the weak upper hemicontinuity of $\tilde{\psi}_f(\cdot, t)$ we observe that $\tilde{\psi}_f(p, t)$ is weakly compact for every $p \in \Delta$ and so it is enough to show that $Gr_{\tilde{\psi}_f(\cdot, t)}$ is closed. Pick $(p_n, v_n) \in Gr_{\tilde{\psi}_f(\cdot, t)}$ and assume that $p_n \rightarrow p_0$ and $v_n \rightarrow v_0$ for some $p_0 \in \Delta$ and $v_0 \in E$. Let us first consider the case in which $t \in C_f(p_0)$. By definition, this means that $\tilde{\xi}(p_0, t) \succ_t f(t)$ and so, by the continuity of $\tilde{\xi}(\cdot, t)$, for a sufficiently large n we will have that $t \in C_f(p_m)$ whenever $m > n$. It follows that $e(t) - v_m \in \tilde{\xi}(p_m, t)$ for all $m > n$ and so, by the continuity of $\tilde{\xi}(\cdot, t)$, $e(t) + v_0 \in \tilde{\xi}(p_0, t)$ and $v_0 \in \tilde{\psi}_f(p_0, t)$ as wanted. With a similar argument we can show that $v_0 \in \tilde{\psi}_f(p_0, t)$ when $t \notin D_f(p_0)$. Last, let us consider the case $t \in D_f(p_0) \setminus C_f(p_0)$. In this case it is enough to observe that $v_n \in (\{e(t)\} - \tilde{\xi}(p_n, t)) \cup \{0\}$ for every $n \in \mathbb{N}$ and so v_0 , which is the weak limit of the sequence, must lie in $\tilde{\psi}_f(p_0, t)$.

To prove point (3) we fix a $p \in \Delta$. By the observations made to prove point (1), we know that $\xi(p, \cdot)$ has a measurable graph and that it satisfies all the assumptions of Aumann's selection Theorem (Aliprantis and Border, 2006, Theorem 18.26). This means that there exists an integrable $h_p: T \rightarrow E$ such that $h_p(t) \in \xi(p, t)$ for almost every $t \in T$. But then the relation $g_p: t \mapsto e(t) - h_p$ defines the desired function. $\blacksquare$

Proposition A.7 (3.4) *Let f be an allocation that is not competitive. Then there is a Walrasian objection to f at a price system p if and only if E_+ intersects the set:*

$$\tilde{\Psi}_f(p) := \int \tilde{\psi}(p, t) dt.$$

Theorem A.8 (3.6) *Suppose that (T, Σ, μ) is a saturated measure space and that f is a feasible allocation that is not competitive. Then there exists a Walrasian objection to f .*

Proof. Since f is non-competitive, it is enough to show that $\tilde{\Psi}_f$ meets all the three conditions of Theorem 3.1. This will give us the existence of a price system $p \in \Delta$ such that $\tilde{\Psi}_f \cap E_+ \neq \emptyset$ and so our claim will follow directly from Proposition 3.4.

By Lemma A.6, we know that for every $p \in \Delta$ the correspondence $\tilde{\psi}_f(p, \cdot): T \rightarrow E$ has a measurable graph. We also know that for almost every $t \in T$ the correspondence $\tilde{\psi}_f(\cdot, t): \Delta \rightarrow E$ is weakly upper hemicontinuous. Last we observe that $\tilde{\psi}_f(p, t) \subseteq X(t)$ for every $p \in \Delta$ and $t \in T$, which means that $(\tilde{\psi}_f(p, \cdot))_{p \in \Delta}$ is a well dominated family of correspondences. All the requirements in 3.3 are met and so $\tilde{\Psi}_f: \Delta \rightarrow E$, which is defined as the integral $\int \tilde{\psi}_f(\cdot, t) dt$, is weakly upper hemicontinuous and has non-empty, convex and weakly compact values.

There is only left to prove that for every $p \in \Delta$ there exists a $v \in \tilde{\Psi}_f(p)$ such that $p \cdot v \geq 0$. Fix a $p \in \Delta$ and observe that, by point (3) in Lemma A.6, there exists a $g_p \in \mathcal{L}_X$ such that $g_p(t) \in \psi_f(p, t)$ for every $t \in T$. This means that $p \cdot g_p(t) \geq 0$ for almost every $t \in T$ and so $p \cdot \int g_p(t) dt \geq 0$. Since $\psi_f(p, t) \subseteq \tilde{\psi}_f(p, t)$ for every $t \in T$, we know that $\int g_p(t) dt \in \tilde{\Psi}_f(p)$ and so it is enough to put $v = \int g_p(t) dt$. ■

Theorem A.9 (3.7) *Suppose that (T, Σ, μ) is a saturated measure space. Then:*

$$\mathcal{W}(\mathcal{E}) = \mathcal{C}(\mathcal{E}) = BS(\mathcal{E}).$$

Proof. We know that $\mathcal{W}(\mathcal{E}) \subseteq \mathcal{C}(\mathcal{E}) \subseteq BS(\mathcal{E})$ by Proposition 2.7. It is therefore sufficient to show that $BS(\mathcal{E}) \subseteq \mathcal{W}(\mathcal{E})$.

To see this, take a feasible $f \in \mathcal{L}_X$ that is non-competitive and observe that, by Theorem 3.6, there exists a Walrasian objection to it. But every Walrasian objection is justified by Theorem 2.10, which means that f does belong to $BS(\mathcal{E})$. ■

Theorem A.10 (3.8) *Suppose that (T, Σ, μ) is a saturated measure space. Then there exists a competitive allocation.*

Proof. Let us focus on the initial endowment e . If e is competitive there is nothing to prove so assume that $e \notin \mathcal{W}(\mathcal{E})$. By Theorem 3.6 there exists a $(S, g) \in Ob(e)$ that is Walrasian at some $p \in \Delta$. This means that the assignment $\hat{e}_{S,g}$ induced by (S, g) , beside being obviously feasible, is supported by p . From Lemma 2.3 we can conclude that $\hat{e}_{S,g}$ is a competitive allocation. ■

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