Behavioral Economics of
Crime Rates and Punishment Levels

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Abstract. Empirical studies have shown, paradoxically, that increasing the probability of apprehension can correlate with an increase in the total number of criminal actions. To examine this phenomenon, this paper develops a theory of "personal rules" based on the tradeoff between one’s self-image of criminal productivity and the temptation – salience of the present – of taking the easy way out by committing a crime. This theory analyzes transformation of lapses into precedents that undermine future self-restraint. The foundation for this transformation is imperfect recall of one’s own criminal productivity within certain defined parameters, which leads people to draw inaccurate inferences from their past actions. Rationalization may lead to overestimation of the expected utility of committing a crime when the opportunity presents itself.

Keywords: Crime, Imperfect Recall, Willpower.

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1 Introduction

If a government’s objective is to minimize crime rates, two questions naturally arise: First, what are the costs and benefits an agent takes into consideration when deciding whether to commit a crime? Second, if resources are limited, how much should the government spend in order to minimize crime rates?

Becker’s (1968) seminal paper on the theory of criminal behavior argues that if criminals are rational, a higher probability of apprehension or higher fines as punishment will lead to a fall in the number of crimes. Because apprehension is a costly activity to the state, and the fine is a costless transfer from the criminal to the state, Becker recommended that the state should set the fine at its highest value. The higher probability of apprehension, Becker asserted, complements the higher fine in deterring individuals from committing a crime. In Becker’s theory and in many observations, criminal activity is monotonic with the probability of apprehension.

The prediction by Becker (1968) is not always consistent with the empirical literature. Myers (1983) studied one year follow-up data for offenders released from US prisons in 1972 and empirically showed that "increases in the certainty of punishment are positively related to participation in crime." This paradox is also found in a number of empirical studies about the relationship between the certainty of punishment (and, therefore, apprehension) and deterrence. Tittle and Rowe (1974) analyzed data for Florida cities in 1970 and found that certainty of arrest was positively related to crime rates if certainty of arrest was below 30%. Fagan and Meares (2008) suggested a paradox of punishment rates versus criminal activity, especially among minorities in the US, where harsher punishment corresponds with counterdeterrent effects on crime rates.

Aside from the above mentioned papers, other literature in criminology also suggested the non-monotonicity of criminal activity with the probability of apprehension, in contrast to Becker’s theory.1 This paper proposes an alternative theory to Becker’s that can be reconciled with these counter-intuitive, empirical findings.

To examine the monotonicity or non-monotonicity of apprehension versus criminal action, this paper assumes that a criminal, DM, has two opportunities for criminal action in period 1 and period 2. He is endowed with criminal productivity $v$

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1Katz, Levitt, and Shustorovich (2003) analyzed state level panel data and found the non-monotonic relationship between criminal behavior and execution rates.

According to a survey conducted by Dieter (1995), around two-thirds of the police chiefs and county sheriffs did not believe that death penalties are effective in reducing homicidal rates.
which defines how much he earns by committing a criminal act in each period. Before any decision to act or not, DM gets a noisy, undefined signal about \( v \); this is soft information. This information about his criminal productivity \( v \) remains soft at period 2 if no criminal action was taken in period 1.\(^2\) For example, the signal could be the fleeting impression about \( v \) DM obtains from an acquaintance who is trying to recruit DM. Reder (1996), Trope (1978), Tulving and Thomson (1973), and Tversky and Kahneman (1971) all demonstrated imperfect recall of soft information.

The probability of a criminal action occurring in period 2 is monotonic with the probability of apprehension if, and only if, DM chose criminal action in period 1 and has hard evidence (incarceration or keeping the stolen property) concerning the productivity of his criminal action. The presence of this hard evidence allows DM to make a decision concerning criminal action in period 2 that is largely monotonic with the probability of apprehension.\(^3\)

If, however, DM does not commit a criminal act in period 1, DM will only be able to recall the signal imperfectly because he has been leading an honest life as a regular worker and hence does not need to access this signal during period 1, leaving him with only a vague impression of it by period 2. His decision at the second opportunity for crime is then based on conjecture or inference about \( v \) (how successful he could have been in period 1) rather than on hard evidence of \( v \) (how successful was he in period 1), as a result of imperfect recall taking effect. In this paper, the concept of imperfect recall plays a crucial role in affecting the monotonicity between the probability of apprehension and criminal action.

Imperfect recall has been given a lot of attention in the studies of strategic decision making in the recent years. Piccione and Rubinstein (1997a) first identified the importance of imperfect recall and showed its influence on decision-making. Subsequently, many papers explored this concept. Piccione and Rubinstein (1997b), Ambrus-Lakatos (1999), and Kline (2002) analyzed the effect of imperfect recall on human behavior in extensive games. We also examine how imperfect recall affects decision making in a multiple period model. However, unlike in their models, im-

\(^2\)The idea of a distinction between hard information and soft information was used by Mullainathan (2002), Bernheim and Thomsdson (2005) and Bénabou and Tirole (2004, 2009). Soft information does not leave a material record and is difficult to verify later. We discuss these works in further detail in Section 2.

\(^3\)We claim that given DM commits a crime in period 1, the probability that he commit a crime period 2 decreases with the probability of apprehension. This claim of (conditional) monotonicity is supported in Polinsky and Shavell (1998), Chu, Hu, and Huang (2000), Emons (2003), and Mungan (2010).
perfect recall does not affect an individual’s beliefs about other players in our model. Rather, our model involves only one player, where the effects of imperfect recall are related to DM’s self-belief or self-inference.

In our model, DM commits a crime in period 1 when his signal (about his criminal productivity $v$) is above a certain threshold value in period 1. Suppose DM does not commit a crime in period 1. In period 2, he imperfectly recalls his signal and hence has to infer $v$ based on the threshold in period 1. In this case, he commits a crime in period 2 if his inferred $v$ is above a threshold in period 2. The threshold in each period increases with the probability of apprehension. Hence, his inferred $v$ in period 2 increases with the probability of apprehension. As a result, for a certain range of the probability of apprehension, his inferred $v$ can exceed the threshold in period 2, causing DM to commit a crime in period 2 although he did not in period 1.\(^4\) Conditional on that DM does not commit a crime in period 1, his criminal action in period 2 can be non-monotonic with the probability of apprehension.

The non-monotonicity between the probability of apprehension and criminal actions is not just created by this imperfect recall of soft evidence concerning criminal productivity alone. For this non-monotonicity to take place, imperfect recall of soft evidence has to be complemented with the hyperbolic discounting of payoffs from a criminal action (DM has present-biased and time-consistent preferences).\(^5\) At the moment when DM decides whether he commits a crime in each period, his temptation leads him to hyperbolically discount criminal payoffs: he magnifies the immediate benefits of a criminal action compared to the (non-immediate) costs of a criminal action.

Criminologists have supported the view of criminals being more present-biased than non-criminals. Gottfredson and Hirschi (1990) proposed that criminal actions

\(^4\)See Section 5 for more information.

\(^5\)The following papers suggest that non-monotonicity can also be influenced by the DM’s personality, specifically, his powers of self-control under circumstances of short-term versus long-term gain.

The "Stanford marshmallow experiment" demonstrates that children who could delay gratification for a larger reward at ages 3-5 developed into adolescents who "were significantly more competent" and their ability to delay gratification at age 3-5 correlated with higher SAT scores in high schools. This experiment with young children demonstrates the effect of hyperbolic discounting on choices for future well-being and suggest this variable ability to delay gratification can contribute to a non-monotonic relationship between criminal activity and the probability of apprehension (Mischel et al., 1972; Shoda et al., 1990).

Lee and McCrary (2005, 2009) used a large sample on felony arrests in Florida and found that hyperbolic discounting affects offenders’ decisions on criminal actions in predicting criminal behaviors or success in life.
are linked to robust individual-level traits such as low self-control. Gottfredson and Hirschi asserted that "people who lack self-control will tend to be impulsive, insensitive, physical (as opposed to mental), risk taking, short-sighted, and nonverbal, and they will tend therefore to engage in crime and analogous acts." This hypothesis has gained substantial empirical support, and attention has been devoted to testing the major components of this self-control theory. Consistent with Gottfredson and Hirschi, a variety of studies have found self-control to be important in predicting criminal behavior.\(^6\) Criminal behavior is closely related to hyperbolic discounting of criminal payoffs.

The body of the paper is organized as follows: Section 2 presents related literature. Section 3 provides the model’s setup. Section 4 explains important assumptions in our model and relates these assumptions to empirical evidence and current literature in criminology. Section 5 analyzes the model. Section 6 discusses why imperfect recall and hyperbolic discounting are both required to obtain a non-monotonic relationship between the probability of apprehension and the total number of criminal actions. Section 7 considers results of the model when the assumptions made in Section 3 are relaxed. Section 8 concludes. Proofs are gathered in the Appendix.

### 2 Related Literature

**Inclusion of behavioral factors in classical analysis of criminal behavior.**—In the recent years, many studies in criminology have identified the limitations of the classical approach towards analyzing criminal behavior. Jolls, Sunstein, and Thaler (1998) suggested a modification of the classical model through the inclusion of behavioral concepts such as "bounded rationality, willpower and self-interest" to achieve more realistic results. Jolls and Sunstein (2005) emphasized the need to address the existence of bounded rationality in individuals in order to improve the existing legal system.

Our paper also aims to address the limitations of the classical model. Like Polinsky and Shavell (1998), we adopt a two-period model approach which takes into consideration the effects of "prior convictions" on criminal behavior. Like Garoupa (2003), we recognize the merit of the classical approach.\(^7\) Hence, instead of deviating

\(^6\)For example, low self-control is significantly related to drunken driving (Piquero and Tibbetts, 1996), official-delinquency (Wood et al., 1993), and adult criminal and imprudent behaviors (Burton et al., 1994).

\(^7\)Garoupa (2003) found that the pitfalls in the behavioral models in criminology exceed that of
from the classical model, we retain it by modifying it.

As stated in the introduction, the two main behavioral factors in our model are hyperbolic discounting and imperfect recall. McAdams and Ulen (2008) stated that the absence of these two factors in the classical model is one of the main reasons why its predictions are not always consistent with empirical studies. Indeed, with the inclusion of these two factors, our paper produces results that match findings in the empirical studies.8

Personal Rules.—People are aware of their own tendencies to exhibit present-bias and hence rely on cognitive rules to prevent themselves from giving in to such temptation. The application of personal rules in decision-making has been studied repeatedly.9 The current literature points out that people are aware of their own proclivity towards present consumption and hence are willing to use cognitive rules to offset the ill-effects of such preferences. According to Wertenbroch (1998), people impose personal rules on themselves in the form of pre-commitment so that a decision made during some period eliminates the possibility of them engaging in myopic behavior during the next.10 Carrillo and Mariotti (2002) also stated that people make an effort to abstain from information about undesirable activities to prevent themselves from engaging in them.

For the concept of personal rules, Bénabou and Tirole (2004) is most closely related to our paper as the foundation of their model is imperfect recall and hyperbolic discounting. They stated that due to the presence of imperfect recall, cognitive rules allow an individual to endogenously determine his memory and hence avoid his myopic behavior.11 Thus, imperfect recall offsets the (negative) effect which hyperbolic discounting has on individuals’ decision-making. On the other hand, the effects of imperfect recall and hyperbolic discounting do not offset each other in our model. Our primary result (the non-monotonicity between the probability of apprehension and criminal actions) is observed only when imperfect recall and hy-

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8See Section 5 for more information.
10For instance, a person who is on a diet would deliberately make the choice of going to food places that only serve non-fattening food, to prevent himself from ordering fattening foods when he is required to make a food decision.
11Bénabou and Tirole (2004) stated, "The fear of creating precedents and losing faith in oneself then creates an incentive that helps counter the bias toward instant gratification."
perbolic discounting complement each other in affecting individuals’ decision making.\textsuperscript{12} Nevertheless our findings do not contradict with theirs. On the contrary, one of our findings actually does support the notion of criminals using personal rules through exerting self-control on themselves.\textsuperscript{13}

\textit{Optimal Punishment}.—The optimal punishment level of criminals has always been the controversial subject of interest in criminological studies. In particular, there has been discourse on the optimal punishment level for repeat offenders. Polinsky and Shavell (1998), Chu, Hu, and Huang (2000), and Mungan (2010) endorsed the view that repeated offenders should be punished more heavily than first-time offenders while Emons (2003) did not entirely support this view.

Polinsky and Shavell (1998) proposed an alternative model to the standard economic model of deterrence.\textsuperscript{14} They adopted a two-period model approach, which is similar to our model, and showed that the effect of deterrence can be enhanced by punishing repeat offenders more heavily than first-time offenders. Chu, Hu, and Huang (2000) stated that repeated offenders should not be given the benefit of doubt like the first-time offenders.\textsuperscript{15} Mungan (2010) claimed that heavier punishment is required for repeat offenders than first-time offenders because repeat offenders are better able to avoid apprehension than first-time offenders.\textsuperscript{16} On the other hand, Emons (2003) showed that increasing punishment levels for first-time offenders is generally more effective in deterring crime than increasing punishment levels for the repeat offenders because the first offenders have larger wealth to lose due to apprehension than the repeat offenders.\textsuperscript{17}

Our main focus is on first offenders, but the implication of our results on repeat offenders does not contradict with the above mentioned papers. In our model, if DM did not commit a crime in period 1, he (a potential first offender) may be more inclined to commit a crime in period 2 as the probability of apprehension increases (non-monotonicity). On the other hand, if he committed a crime in period 1, he

\textsuperscript{12}We will further elaborate on the interaction between imperfect recall and hyperbolic discounting in Section 6.

\textsuperscript{13}For more information, see Proposition 3 in Section 5.

\textsuperscript{14}According to Polinsky and Shavell (1998), the standard model of deterrence does not sufficiently address the effects of a prior criminal experience of an individual on his crime tendency.

\textsuperscript{15}Chu, Hu, and Huang (2000) take into account the “possibility of the erroneous conviction of innocent offenders” and hence propose a more lenient punishment for first-time offenders.

\textsuperscript{16}Mungan (2010) found that repeat offenders learn from past experiences and hence become more adept at avoiding apprehension during their subsequent criminal attempts.

\textsuperscript{17}According to Emons (2003), increasing a punishment level for the repeat offenders is efficient only if criminals have sufficiently large wealth.
(a potential repeat offender) is less inclined to commit a crime in period 2 as the probability of apprehension increases (monotonicity). Hence, our paper ultimately suggests that the net effect of an increase in the probability of apprehension on crime rates is largely dependent on the demographics of the criminal population.18

**Self-signaling.**—People often learn about themselves by observing their past actions. Conversely, they often make choices to preserve favorable self-images. This is well documented in psychology research.19

Like our paper, Bénabou and Tirole (2006b) predicted a positive correlation between the probability of apprehension and criminal actions.20 Their assumptions of factors influencing the decision are more complex than our assumptions. For example, their DM considers monetary payoffs, values of his self-image and values of social reputation about his altruism. Their study indicates that the lower probability of apprehension may increase the value of his self-image or social reputation as a result of avoiding a crime and, therefore, result in a smaller number of criminal actions. On the other hand, our paper assumes that DM only considers monetary payoffs. We show that the lower probability of apprehension may decrease DM’s confidence in his criminal productivity, which may result in a smaller number of criminal actions.

Akerlof and Dickens (1982) and Dickens (1986) predicted non-monotonicity between severity of punishment and a number of criminal actions using a non-Bayesian framework that assumes DM directly chooses his belief about the value of the crime to avoid a psychic cost of cognitive dissonance.21 On the other hand, our study predicts non-monotonicity between the probability of apprehension and a number of criminal actions using a Bayesian framework that assumes DM does not directly choose his future inference. DM’s prior action affects his information, as discussed earlier, which affects his present inference.

Recent research suggested that non-monotonicity can be influenced by self-confidence and malleable beliefs.22 In Bénabou and Tirole (2004), DM who has greater confid-

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18See Section 5 for more information.
19In a well-known experiment, Quattrone and Tversky (1984) showed that subjects who were led to believe that their tolerance for a certain kind of pain (keeping one’s hand in very cold water) was diagnostic of either a good or a bad heart condition reacted by extending or shortening the amount of time they withstood that pain, respectively.
20Precisely, Bénabou and Tirole (2006b) showed a negative relationship between prosocial actions and the probability that such actions are observed by others.
21In Akerlof and Dickens (1982) and Dickens (1986), DM incurs a psychic cost when he commits a crime while he expects huge losses from criminal actions, or when he does not commit a crime while he expects large benefits from criminal actions.
22Soman (2001) and Wertenbroch et al. (2001) provided empirical support for memory management
ence in his willpower will exercise self-control more effectively and avoid future mistakes. In our paper, greater confidence in DM’s criminal productivity weakens the effectiveness of self-control and increase future mistakes since it increases the temptation DM faces.

In Bénabou and Tirole (2009), malleable beliefs such as forgetting soft information may make DM worse off. Because of malleable beliefs, DM may inaccurately overestimate his productivity. As a result, DM fails to select an optimal action. In this paper, malleable beliefs may also make DM better off. Because DM knows that he will be tempted to commit a crime in period 2, he is willing to not commit a crime in period 1 in order to protect himself from the temptation he will face in period 2.

The type of information.—The idea of imperfect recall is widely supported by empirical studies discussed in the introduction (Reder 1996, Trope 1978, Tulving and Thomson 1973, and Tversky and Kahneman 1971). In addition, our model is closely related to theoretical works of Mullainathan (2002), Bernheim and Thomadsen (2005) and Bénabou and Tirole (2009), all of whom assumed that memorability of information depends on the type of information. "Hard information," which leaves a material record, is perfectly recalled since it is verified later. "Soft information," which does not leave verifiable records, may not be recalled perfectly. Past criminal actions and income from criminal acts constitute hard information since they are verified later. DM’s impression of how much he thinks he can earn committing a criminal act is soft information as it does not leave a verifiable record.

Mullainathan (2002) assumed that people recall soft information more perfectly if it is rehearsed. However, if a person chooses to spend an honest working life away from events that remind him of the past, his past impression of criminal productivity, \( v \), is not perfectly recalled. While Mullainathan focused on the naive model where people do not adjust to the fallibility of memory, this paper uses the sophisticated model.

In Bernheim and Thomadsen (2005), people know the limitations of their memory and make direct costly investments to sustain perfect memory in the future. In this model, as in Bénabou and Tirole (2009), people choose actions that affect payoffs today, and these actions affect memory in the future. Hence, when DM chooses a payoff-relevant action, he needs to consider both the marginal effect on immediate payoffs and the marginal effect on future payoffs through the effect on his future by testing consumers’ recall of their recent expenditures.
memory.

The idea of imperfect recall is widely supported by empirical studies in biology and psychology. Structurally, this model is closely related to Mullainathan (2002), Bernheim and Thomadsen (2005) and Bénabou and Tirole (2009), all of whom assume that memorability of information depends on the type of information.

Hyperbolic discounting.—A branch of psychology has been devoted to understanding behaviors characterized by strong internal conflicts and harmful impulses that cause an individual to succumb and act against better judgment.\(^{23}\)

Experimental psychologists have documented this robust feature of time inconsistent preferences that commonly gives rise to self-control problems, namely, people’s tendency to discount payoffs much more steeply at long than at short horizons.\(^{24}\)

Criminologists have suggested that criminal actions are essentially associated with impulsivity. Criminal actions yield easy and immediate gains, and do not require waiting for benefits from complex tasks or tedious works (Gottfredson and Hirschi 1990).\(^{25}\) When offered easy and immediate gratification from criminal actions, some people magnified the costs of honest work in decision making.

Although Becker’s theoretical model draws a monotonic relationship between the probability of apprehension and a decrease in criminal activity, the existing literature presents a number of other factors, including self-signaling, imperfect recall, impulsivity, and hyperbolic discounting, that can all influence a DM’s decision about criminal actions and thus distort the statistical monotonicity expected between the probability of apprehension and the probability of criminal activity.

3 Model

This game considers a decision maker (DM) with a horizon of two periods, \(t=1, 2\). At each period \(t\), DM can either commit a crime \((a_t=1)\) or not \((a_t=0)\).

\(^{23}\)In Bénabou and Tirole (2004, 2006a), the individual is faced with a choice between a course of action that requires no self-restraint and another that challenges his capacity to resist temptation and hold out for larger and long-run payoffs.

\(^{24}\)Ainsley and Haslam (1992) argued that humans share a general preference for immediate gains over future gains, which may cause hyperbolic discounting. Pratt and Cullen (2000) conducted a meta-analysis on existing empirical studies and suggest impulsive behaviors are associated with criminal acts.

\(^{25}\)Frank (2005) and Utset (2007) argued that impulsivity is essentially connected with criminality, and crime deterrence can be improved by accounting for the effects of hyperbolic discounting. Kahneman et al. (1997) argued that hyperbolic discounting is likely accountable for drug use.
Before the start of the game, DM is endowed with criminal productivity $v$, which represents how much he earns by committing a crime. The productivity $v$ is drawn from an exponential distribution with parameter $\lambda=1$.

At $t=1$, DM receives an informative but noisy signal, $\sigma$:\n\begin{equation}
\sigma = v + \varepsilon
\end{equation}
about his criminal productivity $v$, where $\varepsilon$ is drawn from an exponential distribution with parameter $\lambda=1$, and $\varepsilon$ and $v$ are independently and identically distributed.

Let $f_{v,\sigma}(\cdot, \cdot)$ denote the probability distribution function (p.d.f.) of DM’s criminal productivity $v$ and the signal $\sigma$ that he receives. We will denote DM’s information at the beginning of each period $t$ by $\Omega_t$. DM’s information at $t=1$ is expressed by $\Omega_1=\sigma \in [0, 1]$.

After observing a signal, DM chooses the probability of committing a crime ($a_1=1$), which is given by $\mu_1 \in [0, 1]$.

If DM commits a crime ($a_1=1$), DM will be apprehended with probability $p \in (0, 1)$. DM knows the probability of apprehension $p$. If he is not apprehended, DM will have the payoff $v>0$, where the payoff depends on his criminal productivity $v$. If he is apprehended, he will be fined $F>0$.

If DM does not commit a crime ($a_1=0$) but chooses to work, he will receive wage income $W>0$ in the end the period.

This paper makes a distinction between "hard information," which leaves a trace of hard evidence, and "soft information," which does not leave such evidence. In this model, hard information includes his action $a_t$, and the payoff $v$ he earns when he

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26The probability density function of the exponential distribution is $\lambda e^{-\lambda v}$ for $v \geq 0$ and 0 for $v < 0$, where $\lambda = 1$.

27For example, DM was in a bar, and was recruited to commit a street crime by a street gang member. The 50-year-old mother, who has lived in Englewood, talked about how gangs in the area had recruited her son: "Every time they see him, they ask him if he wants some money. ... It starts with little things like that, and before you know it, it will escalate to bigger things." [Chicagodefender.com, July 9, 2008].

28$\mu_t=1$ ($\mu_t=0$) means that DM uses the pure strategy in period $t$. $\mu_t \in (0, 1)$ means that DM uses a mixed strategy in period $t$. This model needs to consider mixed strategies because there is a unique PBE where DM uses a mixed strategy at $t=2$, as shown in Section 5.

29For example, DM is in a bar, and is recruited to commit a street crime by a street gang member. Then, $a_1=1$ means that DM takes the offer, and $a_1=0$ means that DM turns down the offer.
commits a crime. Soft information includes his signal $\sigma$ at $t=1$.

We assume that DM perfectly recalls hard information, and imperfectly recalls soft information under some conditions. If DM commits a crime at $t=1$ ($a_1=1$), he discovers $v$ and perfectly recalls it. For simplicity, apprehension does not affect his memory of $v$. On the other hand, if DM does not commit a crime at $t=1$ ($a_1=0$), he will not discover $v$ and completely forget $\sigma$.

Let $\hat{\sigma}$ denote DM’s memory related to his criminal productivity at $t=2$, where $\hat{\sigma}=v$ if $a_1=1$ and $\hat{\sigma}=\emptyset$ otherwise. Thus, DM’s information at $t=2$ is $\Omega_2=(\hat{\sigma}, a_1) \in [0,1] \cup \{\emptyset\} \times \{1,0\}$. Figure 1 illustrates this path-dependent information.

At $t=2$, DM chooses the probability of committing a crime again. The probability is denoted by $\mu_2 \in [0,1]$. The costs and benefits associated with his action are the same as at $t=1$. The difference from $t=1$ is his information, which is $\Omega_2=(\hat{\sigma}, a_1)$.

This paper assumes that DM’s preference exhibits time inconsistency. Thus, at each of $t=1,2$, when DM makes a decision, he is tempted or experiences “salience of the present” at the thought obtaining “easy” money instantly through criminal activities instead of working hard.

Criminal action yields immediate gain, $v$, to DM, while the other benefits and costs such as the wage income, $W$, and the fine, $F$, are realized in the end of the period. Hence, when DM makes a decision, he discounts the delayed benefits and

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30 The knowledge of his criminal productivity $v$ that he obtains by committing a crime is hard information discovered through monetary gain, which is verifiable evidence. In addition, his past action $a_1$ is also hard evidence. If he commits a crime, he goes to the street and earns from stealing an unlocked car. If he works honestly, he goes to his workplace every day and receives a salary. Either action leaves hard evidence.

31 The signal, $\sigma$, is his impression about his criminal productivity, $v$, which he obtains through transient interactions with other people. Hence, this is soft information, which is not verified later.

As a real life example, there are different perceptions of how profitable auto theft is. In Washington, D.C., a witness told a Senate committee that he became a car thief because “it was very easy and there was big money in it.” [Milwaukee Journal, Nov.28, 1979] On the other hand, an anonymous reader left comments on car theft that “a pretty dumb idea…” [Associated Press, Jan. 25, 2012]

32 Realistically, if DM does not commit a crime, he should recall his signal with some positive probability. Since adding this complication does not lead to qualitative differences in the results, this paper studies the simplest case.

33 In this case, one week later DM thinks whether to take the past offer to commit a crime. For example, $a_2=1$ means that DM goes back to the bar and takes the offer. $a_2=0$ means that DM does not go to the bar, and hence does not take the offer.

34 This means that DM does not collect additional information in period 2. Realistically DM can go back to the bar and collect an additional signal. Since adding this complication does not lead to qualitative differences in the results, this paper considers the simplest case.

costs; equivalently, he values the immediate gratification from a criminal action at \( v / \beta_1 \) instead of \( v \) but values \( W \) and \( F \) as they are in each period, where \( \beta_1 \in (0, 1] \) denotes a rate of hyperbolic discounting.\(^{36}\)

We assume \( \beta_1 > \beta_2 \), where smaller \( \beta_t \) means stronger temptation.\(^{37}\) DM knows the level of his temptation in every period. Since DM in this case has no way to resist his temptation at \( t=1 \), we assume \( \beta_1 = 1 \).\(^{38}\) We will explain the connection between this assumption and empirical evidence in Section 4.

DM’s utility perceived by DM at \( t=1 \), denoted by \( U_1 \), is given by:

\[
U_1 = E \left[ \mu_1 \cdot \left( \left( 1 - p \right) \cdot \frac{v}{\beta_1} - p \cdot F - W \right) + W | \sigma \right].
\]

Likewise, DM’s utility perceived by DM at \( t=2 \), denoted by \( U_2 \), is given by:

\[
U_2 = E \left[ \mu_2 \cdot \left( \left( 1 - p \right) \cdot \frac{v}{\beta_2} - p \cdot F - W \right) + W | \Omega_2, \mu_1 \right].
\]

The sequence of events is shown on the timeline in Figure 2.

### 4 Assumptions of Model

This section explains the connection between important assumptions in our model and the relevant research papers.

\(^{36}\)This form of hyperbolic discounting is closely related to that of Bénabou and Tirole (2004, 2006a). Due to this form of imperfect willpower, DM will commit a criminal act even though it is unprofitable from an ex-ante standpoint. For example, if DM commits a crime, he can earn a lot of money by stealing an unlocked car for a few hours. However, if DM works honestly, he needs to complete a week’s worth of hard work like mopping the floor and throwing out the trash before getting paid.

\(^{37}\)Hoch and Loewenstein (1991), Shiv and Fedorikin (1999), Hinson, et al. (2003) and Vohs and Faber (2007) studied consumer behavior and empirically show that temptation can be increasing over time.

For example, in our model, DM is new to the workforce at \( t=1 \). After working for some time, DM has realized how difficult it is to make an honest living. For this reason, the temptation he faces tomorrow is stronger than it is today. One day in the past DM was recruited to commit a street crime. DM was tempted to earn a lot of money by stealing an unlocked car for a few hours. One week later he remembers the past offer, and is more tempted to take the offer.

\(^{38}\)All the results remains unchanged given \( \beta_1 < 1 \).
One of important assumptions is $\beta_1 \geq \beta_2$, which implies that crime temptation increases with time whether or not DM committed a crime in the previous period.

We first consider the case $a_1=1$, meaning that DM committed a crime at $t=1$. We introduce empirical findings pertaining to drug usage because a vast majority of the general criminal population abuse drugs.\(^{39}\) Volkow (2014) stated, "repeated drug use disrupts well-balanced systems in the human brain in ways that persist, eventually replacing a person’s normal needs and desires with a one-track mission to seek and use drugs." This statement implies that for an individual with prior usage of drugs (i.e., $a_1=1$), his desire to use drugs grows with time.\(^{40}\)

Polinsky and Shavell (1999) claimed that some criminals discount the disutility of the later years of imprisonment more than they do for the earlier years. This claim is also consistent with the assumption $\beta_1 \geq \beta_2$ since a more discounted net benefit necessarily leads to a higher level of crime temptation. Fajnzylber, Lederman, and Loayza (2002) also pointed out the presence of "criminal hysteresis or inertia" in criminals, implying that an individual’s past criminal activities increase his own propensity towards committing a crime during subsequent time periods.

Next, we consider the case $a_1=0$, meaning that DM did not commit a crime at $t=1$. According to studies on drug abuse, curiosity is one of the major reasons why individuals start using drugs, and curiosity-induced temptation towards the first drug use increases with time.\(^{41}\) When an individual is provided with sufficient information regarding the thrills of drug usage, his desire to use drugs lingers and increases until he eventually does.\(^{42}\) These findings regarding the first drug use are consistent with our assumption $\beta_1 \geq \beta_2$ given $a_1=0$.

On the other hand, some empirical findings in criminological studies may appear to contradict with the assumption $\beta_1 \geq \beta_2$. However, upon closer securitization, we find out that our assumption does not contradict these findings. For example, Levitt and Miles (2007) stated, "crime propensities decline with age, as widely observed." This decline in crime propensity is caused by a decline in the individual criminal pro-

\(^{39}\)According to Bureau of Justice Statistics (2004), more than 2/3 of the prisoners in the United states have drug problems.

\(^{40}\)According to National Institute of Drug Abuse (2009), for heroin addicts, “memory of the cocaine experience or exposure to cues associated with drug use can trigger tremendous craving and relapse to drug usage,” after long periods of abstinence.

\(^{41}\)See Plant and Plant (1992) and McIntosh, Macdonald, and Mckeganey (2002) for more information.

\(^{42}\)According to Lord (2003), curiosity is a feeling that doesn’t (just) fall into our lap fortuitously and rest there.
ductivity due to the age-imposed physical limitations. Polinsky and Shavell (2007) also stated, "evidence suggests that harm caused by individuals declines with their age." Again, this statement is associated with increasing physical restrictions with age, rather than a decline in criminal intent. Thus, Levitt and Miles (2007) and Polinsky and Shavell (2007) indicated that criminal productivity $v$ declines with age. Moreover, they considered longer period than we do. Our time period, which is required for temptation towards crime to increase, is significantly shorter than what is required for aging to affect individuals’ criminal productivity.

The second assumption is $\sigma = v + \epsilon$, which implies that DM always overestimates his productivity when only soft information is available to him.

According to the current literature, the proclivity to crime is due to a mixture of psychological and situational factors. Psychologists have shown that the tendency to overestimate one’s own ability is present not only in criminals, but also in the general human population.\textsuperscript{43} Bénabou and Tirole (2009) developed a theoretical microeconomics model involving the salience of an individual’s self-esteem. According to them, the individual’s emphasis on his own dignity affects his ability to make rational decisions. When DM only has soft information (a noisy signal) on his dignity, he is more likely to overestimate his level of dignity as he has stronger salience of self-esteem.

Criminologists have also reported empirical findings that are consistent with the assumption $\sigma = v + \epsilon$. In an inmate survey conducted by Rand Corporation in 1978, most mid-rate crime offenders exaggerated their individual criminal productivity by a huge margin, and this overestimation is caused by their own desire to "justify their impulsiveness (then) to themselves."\textsuperscript{44} Benson (1985) found that white-collar crime offenders do not identify themselves as criminals, but think of themselves as individuals who engaged in activities that breached the law for inexplicable reasons. These findings imply that criminals overestimate their criminal productivity during criminal activities.

The next assumption is that DM does not seek for additional information regarding his criminal productivity to his signal. This assumption may seem counter-intuitive. Based on the rational choice theory, an individual chooses to commit a crime when the benefit from the activity exceeds its cost, and hence DM should try

\textsuperscript{43}In Sherrill (2008), the overestimation of one’s own ability is identified as “self-serving bias” and a “cognitive or perpetual process that is distorted by the need to maintain and enhance self-esteem.”

\textsuperscript{44}See Tremblay and Morselli (2000) for more information.
to gather as much information regarding the crime he intends to commit as possible at any given time period.\textsuperscript{45} On the other hand, some studies in criminology and behavioral economics have indicated this is not always true in various situations.

Consider DM at $t=1$. The best ways for him to gather crime-related information can be to join a criminal organisation. Alleyne and Wood (2010) suggested that individuals who exhibit higher proclivity towards crime are more likely to be recruited into such organisations.\textsuperscript{46} On the other hand, empirical evidence from criminological studies indicates that an individual has limited access to information regarding his criminal productivity, even within criminal organizations. According to Tat-wing (2001), while many criminal organizations coexist within a complex network across such organizations, the existing animosity among criminal organizations makes exchange of information among criminals impossible.\textsuperscript{47} Empirical studies have pointed out that many gang groups are formed because the individual gang members feel threatened by the "prior presence of other gang" (Neilander & Ferguson, 2000) and "fearful of victimisation" (Alleyne & Wood, 2010).\textsuperscript{48} Based on these empirical findings, we assume that DM does not seek for crime-related information at $t=1$ after he observes a signal (soft information).

Consider DM at $t=2$. He does not seek for additional information at $t=2$. Instead, he relies on information (whether it is soft or hard) obtained at $t=1$ when he decides whether he commits a crime at $t=2$. This act of refraining oneself from seeking crime-related information indicates that DM exerts self-control on himself through the application of his personal rule. Bénabou and Tirole (2004) described personal rules as a kind of cognitive measure used by individuals to regulate their future behavior. They said that an individual is aware of his own proclivity towards instant gratification and therefore willing to put himself through a commitment device that serves to protect himself from giving in to such temptation. Carrillo and Mariotti (2000) also showed that individuals may choose to avoid certain available information to prevent themselves from engaging in undesirable activities.\textsuperscript{49} Hence, by framing the

\textsuperscript{45}See Scott (2000) for more information.

\textsuperscript{46}According to Alleyne and Wood (2010), cited interactional theory supports their claim that "gangs select and recruit members who are already delinquent."

\textsuperscript{47}Tat-wing (2001) stated, "It is not uncommon that gangs within the same triad society often fight with each other over a disputed interest or territory."

\textsuperscript{48}Klein (1971) also suggested that the level of cohesion within a criminal organisation is positively correlated to degree of perceived the threat by other criminal organisations. These empirical studies imply that crime-related information is not circulated in criminal organizations.

\textsuperscript{49}Carrillo and Mariotti (2000) provides an example that is closely to related to our analysis of crim-
behavioral concept, personal rules, into the context of criminal behavior, we assume that DM does not seek for crime-related information at $t=2$.

5 Analysis

We adopt a mixed strategy Perfect Bayesian Equilibrium (PBE) as the solution concept of our problem. A PBE of this game is a pair $(\mu_1^*, \mu_2^*) \in [0, 1] \times [0, 1]$ of arguments that maximize equations (2) and (3), meaning that:

1. DM’s decision at $t=1$ is optimal, given his inference of his criminal productivity at $t=2$.

2. At $t=2$, DM infers his criminal productivity using Bayes’ rule that takes into account his action and his action rule at $t=1$.

In equilibrium, DM decides his strategy $(\mu_1^*, \mu_2^*)$ at the beginning of the game. This paper assumes DM’s inference at $t=2$ if he chooses an off the equilibrium path as follows. If DM deviates from his equilibrium strategy and does not commit a crime at $t=1 (a_1=0)$, he will infer his criminal productivity $(v)$ based on his past action $(a_1)$ and action rule $(\mu_1^*)$. If DM deviates from his equilibrium strategy and commits a crime at $t=1 (a_1=1)$, he will discover $v$ perfectly.50

This paper shows the existence of a PBE. We analyze the equilibrium using backward induction. At $t=2$, DM simply considers the direct costs and benefits of committing a crime:

$$\text{Benefits} \left( p \cdot F + W \right) - \frac{(1 - p) \cdot E[v|\Omega_2, \mu_1^*, p]}{\beta_2} \text{ Costs}. \quad (4)$$

To build some intuition, first consider DM’s ex-ante optimal action rule at $t=2$ where DM is free from temptation. DM’s ex-ante optimal action rule at $t=2$ is expressed by the cutoff value:

$$Y^*(p) := \frac{p \cdot F + W}{1 - p} \quad (5)$$

50DM’s inference off the equilibrium path in this model is based on arguments about the absent-minded driver and the forgetful passenger in Aumann, Hart and Perryis (1997a, 1997b), Lipman (1997) and Piccione and Rubinstein (1997b).
such that committing a crime with probability 1 ($\mu_2^* = 1$) is ex-ante optimal if $E[v|\Omega_2, \mu_1^*, p] > Y^*(p)$ holds, and committing a crime with probability 0 ($\mu_2^* = 0$) is ex-ante optimal otherwise.

Now consider DM’s ex-post optimal action rule at $t=2$ where DM is affected by his temptation, $\beta_2 \in (0, 1)$. DM’s ex-post optimal action rule at $t=2$ is now given by:

$$Y(p) := \frac{p \cdot F + W}{1 - p} \cdot \beta_2. \quad (6)$$

such that committing a crime with probability 1 ($\mu_2^* = 1$) is ex-post optimal if $E[v|\Omega_2, \mu_1^*, p] > Y(p)$ holds and committing a crime with probability 0 ($\mu_2^* = 0$) is ex-post optimal otherwise, where $Y(p) < Y^*(p)$. Hence, DM’s temptation leads him to make a wrong decision (DM will commit a crime even though it is not profitable for him to do so) if his ex-post inference falls between the ex-ante cutoff value ($Y^*(p)$) and the ex-post cutoff value ($Y(p)$):

$$E[v|\Omega_2, \mu_1^*, p] \in (Y(p), Y^*(p)). \quad (7)$$

There are two possible outcomes at $t=2$. First, suppose DM committed a crime at $t=1$ (i.e., $a_1 = 1$). He discovers his criminal productivity (i.e., $\hat{\sigma} = v$), and hence simply compares $v$ and $Y(p)$ given by (6). As a result, he commits a crime if $v > Y(p)$ holds, and does not commit a crime otherwise. If (7) holds, temptation leads DM to commit a crime at $t=2$ while it is not optimal ex-ante.

Second, suppose DM did not commit a crime at $t=1$ (i.e., $a_1 = 0$). He forgets his signal ($\hat{\sigma} = \varnothing$), and hence infers $v$ from his action rule at $t=1$ ($\mu_1^*$) and his past action ($a_1 = 0$). As a result, he commits a crime if his inference about $v$ is large enough, that is, if $E[v|\Omega_2 = (\varnothing, 0), \mu_1^*, p] > Y(p)$, and does not commit a crime otherwise.
Next, consider DM’s problem at $t=1$. Let $V_1(a_1, \sigma, \mu_2^*, p)$ denote a value function at $t=1$. The marginal benefit from committing a crime at $t=1$ is given by:

$$V_1(1, \sigma, \mu_2^*, p) - V_1(0, \sigma, \mu_2^*, p) = \left( (1 - p) \cdot E[v|\sigma] - p \cdot F - W \right)$$

where $f_{v|\sigma}$ denotes the pdf of $v$ conditional on $\sigma$, and $\mu_2^*(\emptyset, 0)$ denotes DM’s strategy at $t=2$ on the path $a_1=0$. The first term represents marginal benefits which are realized at $t=1$. This term is monotonically increasing in the signal $\sigma$ and monotonically decreasing in the probability of apprehension $p$. The second term and the third term represent the forgone payoff DM would have obtained at $t=2$ if he had not committed a crime at $t=1$ ($a_1=0$). Whether he gains or loses by committing a crime at $t=1$ depends on how effectively he is able to stop himself from committing a crime at $t=2$. This forgone payoff is monotonically increasing in signal $\sigma$. Thus, the marginal benefit from committing a crime at $t=1$, given by (8), is monotonically increasing in $\sigma$, and hence there is a unique cutoff $X(p)$ at $t=1$:

$$X(p) : \in \{ \sigma \in \mathbb{R}_+ : V_1(1, \sigma, \mu_2^*, p) - V_1(0, \sigma, \mu_2^*, p) = 0 \}$$

such that committing a crime $t=1$ ($a_1=1$) is profitable for DM at $t=1$ if $\sigma > X(p)$; and it is not for DM at $t=1$ otherwise.

$X(p)$ is a solution to a fixed point problem which involves DM’s problems at both periods. To understand this, consider DM’s strategy at $t=2$ again. At $t=2$, DM knows he compared his signal $\sigma$ with the unique cutoff value $X(p)$ when he decided $\mu_1$ at $t=1$. If DM has forgotten his signal ($\bar{\sigma} = \emptyset$), he knows that it is due to a lack of criminal experience ($a_1=0$), and hence infers that his signal was below the cutoff at $t=1$ ($\sigma < X(p)$). DM’s inference about his criminal productivity is given by:

$$E[\nu|\Omega_2 = (\emptyset, 0), \mu_1^*, p] = E[\nu|\sigma < X(p), p] = I(X(p)),$$
where:
\[ I(t) := 1 - \left( \frac{t^2}{2} + t + 1 \right) \cdot \exp(-t). \] (11)

\( I \) is strictly increasing in its element and bounded above.\(^{51}\)

On this path (\( a_1=0 \)), committing a crime at \( t=2 \) (\( a_2=1 \)) is profitable for DM at \( t=2 \) if:
\[ I(X(p)) > Y(p), \] (Perceived benefit)
\[ \text{Inference at } t=2 \] (12)
\[ Y(p), \] (Perceived cost)
\[ \text{Cutoff value at } t=2 \]

and it is not profitable for DM at \( t=2 \) otherwise. DM’s inference about his criminal productivity, \( I(X(p)) \), is interpreted as his benefits from a criminal action perceived by DM at \( t=2 \), given he did not commit a crime at \( t=1 \) (\( a_1=0 \)). The cutoff value at \( t=2 \), \( Y(p) \), is interpreted as the costs of a criminal action perceived by DM at \( t=2 \).

At \( t=1 \), DM uses a pure strategy almost everywhere. He commits a crime (\( \mu^*_1(\sigma)=1 \)) if his signal is above the unique cutoff at \( t=1 \) (\( \sigma>X(p) \)). Otherwise he does not commit a crime (\( \mu^*_1(\sigma)=0 \)).

At \( t=2 \), DM may use a mixed strategy if he did not commit a crime at \( t=1 \). First, suppose he committed a crime at \( t=1 \) (\( a_1=1 \)). He discovered his true criminal productivity \( v \). As a result, he uses a pure strategy such that he commits a crime (\( \mu^*_2(v,1)=1 \)) if \( v>Y(p) \); he does not commit a crime (\( \mu^*_2(v,1)=0 \)) otherwise. Next, suppose that he did not commit a crime at \( t=1 \) (\( a_1=0 \)). He should infer his criminal productivity \( I(X(p)) \) based on his past rule. There are three potential consequences: he commits a crime (a pure strategy) if \( I(X(p))>Y(p) \); he does not commit a crime (a pure strategy) if \( I(X(p))<Y(p) \); he mixes between the two actions if \( I(X(p))=Y(p) \).

**Proposition 1** For any level of probability of apprehension \( p \), there exist a unique PBE.

**Proof.** See Appendix \( \blacksquare \)

The important question is how different levels of \( p \) affect the total number of criminal actions across two periods.

We have shown that DM can stop himself from committing a crime at \( t=2 \) without committing a crime (\( a_1=0 \)) only if \( I(X(p))<Y(p) \). In other words, DM can fail to resist temptation to commit a crime at \( t=2 \) even if he succeeded in resisting at \( t=1 \). As

\(^{51}\)See Proof of Proposition 1 in Appendix for a mathematical derivation of \( I \).
mentioned earlier, $Y(p)$ (the cutoff value at $t=2$) is interpreted as the marginal cost of a criminal action at $t=2$, and $I(X(p))$ (DM’s inference about his criminal productivity on the path $a_1=0$) is interpreted as the marginal benefit of a criminal action at $t=2$ on this path. $Y(p)$ is increasing and convex in $p$. On the other hand, $I(X(p))$ is increasing in $p$ and can be concave in $p$.

The analysis shows that the higher probability of apprehension can sometimes result in more criminal actions. Consider the outcome at $t=2$ when DM did not commit a crime at $t=1$ ($a_1=0$). There are three cases based on the probability of apprehension $p$; $p$ is high, medium or low.

When $p$ is low, he does not commit a crime at $t=2$ ($a_2=0$) if he did not commit a crime at $t=1$ ($a_1=0$) because he infers that he is of very low productivity. In other words, when $p$ is low, DM sets the cutoff value $X(p)$ at a very low level; DM does not commit a crime at $t=1$ when he receives a low signal $\sigma<X(p)$. At $t=2$, he infers that he did not commit a crime because he has very low criminal productivity, which prevents him from committing a crime at $t=2$. That is, $I(X(p))<Y(p)$ holds.

When $p$ is high, he does not commit a crime at $t=2$ ($a_2=0$) if he did not commit a crime at $t=1$ ($a_1=0$) because the probability of apprehension is very high. He sets $X(p)$ at a high value, causing him to infer that he is of high criminal productivity at $t=2$. However, the cost of committing a crime dominates the benefit at $t=2$ because $p$ is very high. Thus, DM does not commit a crime at $t=2$. That is, $I(X(p))<Y(p)$ holds.

On the other hand, when $p$ is medium, DM commits a crime at $t=2$ ($a_2=1$) even if he did not at $t=1$ ($a_1=0$). DM thinks that the fact that he did not commit a crime does not necessarily mean he has low criminal productivity. And, $p$ is only medium. Besides, DM is more tempted to commit a crime at $t=2$ than at $t=1$. Thus DM fails to resist temptation at $t=2$ even if he succeeded to resist at $t=1$. That is, $I(X(p))>Y(p)$ holds.

In summary, given that DM did not commit a crime ($a_1=0$), the number of criminal actions at $t=2$ is not monotonically decreasing in the probability of apprehension $p$. See Figure 4 and 5 for an illustration.

This non-monotonicity is observed when DM’s temptation $\beta_2$ is medium. In other words, there exist cutoff values $B_1$ and $B_2$, where $0<B_1<B_2<1$, such that for $\beta_2 \in (B_1, B_2)$, there is non-monotonicity between the number of criminal actions at

\[52\text{Precisely } I(X(p)) \text{ is convex in } p \text{ for small } p \text{ and concave in } p \text{ for large } p.\]
On the other hand, if his temptation is strong ($\beta_2 < B_1$), he will succeed in resisting temptation at $t=2$ on the path $a_1=0$ ($I(X(p)) < Y(p)$) only when $p$ is high; he will fail to resist ($I(X(p)) > Y(p)$) otherwise. This result is monotonic. If his temptation is weak ($\beta_2 > B_2$), he will succeed in resisting temptation at $t=2$ on the path $a_1=0$ ($I(X(p)) < Y(p)$) for any level of $p$. The result is again monotonic.

For example, consider $\beta_2 = 0.3$, $F = 1.5$ and $W = 0.4$, where $\beta_2 \in (B_1, B_2)$. The cutoff value at $t=1$, $X(p)$, and the cutoff value at $t=2$, $Y(p)$, are both monotonically increasing in $p$. But, conditional on the path $a_1=0$, criminal action at $t=2$ is not monotonically decreasing in $p$.

**Proposition 2** Consider $\beta_2 \in (B_1, B_2)$. There exist $P_1$ and $P_2$, where $0 < P_1 < P_2 < 1$, such that in equilibrium:

1. DM will commit a crime at $t=2$ with a positive probability if he did not commit a crime at $t=1$ for $p \in (P_1, P_2)$.
2. DM will not commit a crime at $t=2$ if he did not commit a crime at $t=1$ for the remaining $p$.$^54$

**Proof.** See Appendix

Lastly, consider the total number of criminal actions for two periods, which is given by:

$$ E[a_1 + a_2] := \int_{\sigma} \int_{\sigma} E[\mu_1^*(\sigma) + \mu_2^*(\Omega_2)|\sigma, v] \cdot f_{\sigma, v}(\sigma, v). $$

Precisely, expression (13) denotes the ex-ante expected total number of criminal actions for two periods. As explained earlier, an increase in the probability of apprehension $p$ can increase the number of criminal actions at $t=2$ for $\beta_2 \in (B_1, B_2)$ while an increase in $p$ decreases the number of criminal actions at $t=1$ for any $\beta_2$. For $\beta_2 \in (B_1, B_2)$, the first effect dominates the second effect. In other words, $E[a_1 + a_2]$ is also non-monotonic in $p$, as shown in Figure 6.

**Proposition 3** Consider $\beta_2 \in (B_1, B_2)$. The total number of criminal actions for two periods is not monotonically decreasing in the probability of apprehension $p$.

**Proof.** See Appendix

$^53$See Proof of Proposition 2 in Appendix for mathematical definitions of $B_1$ and $B_2$.

$^54$See Proof of Proposition 2 in Appendix for mathematical definitions of $P_1$ and $P_2$.  

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There is an interesting observation that the cutoff value at \( t=1 \), \( X(p) \), is sometimes decreasing in his level of temptation, \( \beta_2 \), while his cutoff value at \( t=2 \), \( Y(p) \), is monotonically increasing in \( \beta_2 \), as shown in Figure 7. (Smaller \( \beta_2 \) means stronger temptation.) The result implies that the stronger temptation at \( t=2 \) can result in a reduction of criminal actions at \( t=1 \). As his temptation is stronger (\( \beta_2 \) is smaller), he uses a stricter action rule at \( t=1 \) (\( X(p) \) is higher). In other words, DM exercises self-control at \( t=1 \) because he knows he will be affected by temptation at \( t=2 \).

6 Discussion

There are the two key ingredients in the model: temptation (time inconsistency) and imperfect recall. This section discusses what happens if both or either component is removed. Will the result that the total number of criminal actions is not monotonic in the probability of apprehension still hold?

Consider a model in which there is neither temptation nor imperfect recall. DM is not affected by variations in temptation and perfectly remembers what he observed in the past. If DM finds it profitable to commit a crime at \( t=1 \), he will also find it profitable to commit a crime at \( t=2 \). As the probability of apprehension increases, DM is less likely to commit a crime in every period, yielding a monotonic relationship between the probability of apprehension and the total number of criminal actions.

Suppose that DM is affected by temptation, and has perfect recall. He will be tempted to commit a crime tomorrow even if it is not a profitable decision from the ex-ante standpoint. Hence, even if he does not commit a crime at \( t=1 \), he cannot stop himself from committing a crime at \( t=2 \). As the probability of apprehension increases, the benefit from committing a crime is dominated by the cost of committing a crime in every period. Thus, the total number of criminal actions is monotonically decreasing in the probability of apprehension.

Suppose that DM has imperfect recall, but is not affected by temptation; that is, he forgets the signal about his own criminal productivity if he does not commit a crime, but discovers his criminal productivity if he commits a crime. If DM did not commit a crime at \( t=1 \), he will infer his criminal productivity from his past action and his action rule at \( t=1 \). Without temptation, DM knows that if he did not commit a crime at \( t=1 \), it was not because he was trying to exercise self-control, but because it was unprofitable. If criminal activity was not profitable at \( t=1 \), it will not be profitable at \( t=2 \). As the probability of apprehension increases, DM is less likely to commit a
crime in every period.

Thus, in order to understand the observed non-monotonic relationship between the total number of criminal actions and the probability of apprehension in this paper, both imperfect recall and temptation are indispensable.

7 Robustness

This section explains what would happen if some of the assumptions in the model are relaxed. The first assumption is that the probability of apprehension is constant across both periods. What happens if the probability of apprehension changes across periods? For example, let the probability of apprehension at $t=1$ be twice that at $t=2$. Crime would then be more profitable at $t=2$ than at $t=1$, and DM would be more likely to commit a crime at $t=2$ than in this model. Thus, non-monotonicity between the total number of criminal actions and the probability of apprehension remains unchanged.

The second assumption is that criminal productivity $v$ is drawn from an exponential distribution. Then, the paper has shown that DM’s inference on the path $a_1=0$ is increasing and sometimes concave in $p$. The concavity is one of the key features which lead to non-monotonicity between the total number of criminal actions and the probability of apprehension. The result remains unchanged using any thin tail distribution. For example, let $v$ be drawn from a truncated normal distribution whose support is $(0, \infty)$. The inference is affected by the cutoff value at $t=1$, $X(p)$, and how densely criminal productivity $v$ is distributed below $X(p)$. As $p$ increases, $X(p)$ increases, and the marginal increase in density of $v$ below $X(p)$ diminishes eventually due to thin tails of the distribution. Hence, the inference exhibits concavity with respect to $p$.

8 Conclusion

It is important to understand the relationship between the total number of criminal actions and the probability of apprehension. However, long-held assumption of monotonicity between the probability of apprehension and the criminal actions does not hold under all circumstances. Suppose a government’s objective is to attain the smallest number of criminal actions. At the same time, it cannot increase the probability of apprehension substantially due to resource constraints. This model
suggests that a zero increase in the probability of apprehension can be better than a small increase.

This study will single out an interesting avenue for further research. The main idea presented here is that people have imperfect access to their own abilities and motives, and must therefore infer them from their past actions. This framework can provide the foundation for a theory of personal, professional, or socio-cultural identity as a cognitive investment.\footnote{Bénabou and Tirole (2011) develops a theory of individual and collective moral behavior based on a general model of identity in which people care about "who they are" and infer their own values from past choices.}

Many interesting questions remain, such as the optimal design of contracts or the political economy of reforms when agents have motivated beliefs.
Figure 1: Information structure as a result of action taken

Period 1
- Soft Information
- Action
  $a_1=1$
  (Criminal Action)
  $a_1=0$
  (No Criminal Action)

Period 2
- Hard Information
  - Criminal Productivity $v$ (truth)
  - Past Action

Figure 2: Timeline
Suppose $\beta_2=0.3$, $F=1.5$ and $W=0.4$. $p$ denotes the probability of apprehension. $I(X(p))$ denotes DM’s inference in period 2 given the DM did not commit a crime in period 1. $Y(p)$ denotes a cutoff value in period 2.

Figure 3: DM’s inference in period 2, $I(p)$, and the cutoff value in period 2, $Y(p)$

Suppose $\beta_2=0.3$, $F=1.5$ and $W=0.4$. $p$ denotes the probability of apprehension. $X(p)$ denotes a cutoff value in period 1. $Y(p)$ denotes a cutoff value in period 2.

Figure 4: Probability of apprehension, $p$, and the cutoff value in each period, $X(p)$ or $Y(p)$
Figure 5: Non-monotonicity between the probability of apprehension, \( p \), and the number of criminal actions in period 2 given that DM does not commit a crime in period 1, \( E[a_2|a_1=0] \). Suppose \( \beta_2=0.3 \), \( F=1.5 \) and \( W=0.4 \). \( p \) denotes the probability of apprehension. \( E[a_2|a_1=0] \) denotes the expected number of criminal actions in period 2 given that DM will not commit a crime in period 1.

Figure 6: Non-monotonicity between the probability of apprehension, \( p \), and the ex-ante expected total number of criminal actions in two periods, \( E[a_1+a_2] \). Suppose \( \beta_2=0.3 \), \( F=1.5 \) and \( W=0.4 \). \( p \) denotes the probability of apprehension. \( E[a_1+a_2] \) denotes the ex-ante expected total number of criminal actions across two periods.
Suppose $p=0.4$, $F=1.5$ and $W=0.4$. $\beta_2$ denotes a rate of hyperbolic discounting in period 2. $X$ denotes a cutoff value in period 1. $Y$ denotes a cutoff value in period 2.

Figure 7: Relationship between temptation in period 2, $\beta_2$, and the cutoff value in each period, $X(p)$ or $Y(p)$
References


Appendix

Proof of Proposition 1

In the proof, let $\beta=\beta_2$. Let $\mu_\tau^* (\Omega_t; p)$ denote the equilibrium strategy at period $t=1, 2$ conditional on DM’s information $\Omega_t$ and the probability of apprehension $p$.

Let $f_\sigma (\sigma)$ denote the unconditional pdf of signal $\sigma$, and $f_{v|\sigma}(v|\sigma)$ the pdf of criminal productivity $v$ conditional on signal $\sigma$. From the definition of the joint distribution $f_{v,\sigma}(v, \sigma)$, each pdf is given by:

$$f_\sigma (\sigma) = \begin{cases} \sigma \cdot \exp (-\sigma) & \text{for } \forall \sigma > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and:

$$f_{v|\sigma}(v|\sigma) = \begin{cases} \frac{1}{\sigma} & \text{for } v > 0 \text{ and } \sigma > v \\ 0 & \text{otherwise.} \end{cases}$$

DM’s inference at $t=1$ is:

$$E \left[ f_v (v') \mid \sigma \right] = \frac{f_{v,\sigma} (v', \sigma)}{\int f_{v,\sigma} (v'', \sigma) \, dv''}.$$  \hspace{1cm} (16)

DM’s inference at $t=2$ is:

$$E \left[ f_v (v') \mid \Omega_2 \right] = \begin{cases} 1 & \text{if } v' = v \text{ if } \Omega_2 = (v, 1) \\ 0 & \text{otherwise} \end{cases}$$

and:

$$E \left[ f_v (v') \mid \Omega_2 \right] = \frac{\int_{\sigma': f_\tau^*(\sigma'; p)=0} f_{v,\sigma} (v', \sigma') \, d\sigma'}{\int_{\sigma': f_\tau^*(\sigma'; p)=0} \int_{v': f_{v|\sigma}(v'|\sigma')=0} f_{v,\sigma} (v'', \sigma') \, dv'' \, d\sigma'}$$

if $\Omega_2 = (\emptyset, 0)$.  \hspace{1cm} (18)

We show that there exist functions $X(\cdot), Y(\cdot)$ and $I(\cdot)$, which characterize a unique PBE. Fix any $p$. DM’s optimization at $t=1$ given $\Omega_1=\sigma$ is:
\[ \mu_1^* (\sigma; p) = \begin{cases} 
1 & \text{for } \sigma > X(p) \\
0 & \text{otherwise.} 
\end{cases} \]  

(19)

DM’s optimization at \( t=2 \) given \( \Omega_2=(a_2, \bar{\sigma}) \) is:

\[ \mu_2^* (1, v; p) = \begin{cases} 
1 & \text{if } v > Y(p) \\
0 & \text{otherwise,} 
\end{cases} \]  

(20)

and

\[ \mu_2^* (0, \varnothing; p) \in \begin{cases} 
\{1\} & \text{if } I(X(p)) > Y(p) \\
(0, 1) & \text{if } I(X(p)) = Y(p) \\
\{0\} & \text{if } I(X(p)) < Y(p). 
\end{cases} \]  

(21)

\[ Y(\cdot) \text{ and } I(\cdot) \text{ is given by (6) and (11) respectively. Hence, it suffices to show the existence of a unique fixed point } X \text{ and } \mu_2^* (0, \varnothing; p), \text{ which is a solution to DM’s problem at } t=1 \text{ and his problem at } t=2 \text{ on the path } a_1=0: \]

\[ \begin{aligned} 
(X(p), \mu_2^* (0, \varnothing; p)) \\
: \in \left\{ (x, \mu) \in [0, 1]^2 : (23), (24) \text{ and (25) hold.} \right\}, 
\end{aligned} \]  

(22)

where:

\[ H(p, x) := I(x) - Y(p), \]  

(23)

\[ x = x(p, \mu) := \left( \frac{W + F \cdot p}{1 - p} \right) \cdot \left( 1 + \sqrt{1 - \frac{(2 - \beta) \cdot \beta}{2 - \mu}} \right), \]  

(24)

\[ \mu \in \left\{ 
\begin{array}{l}
\{1\} \text{ if } H(p, x(p, 1)) > 0 \\
\{0\} \text{ if } H(p, x(p, 0)) < 0 \\
\{\mu' \in (0, 1) : H(p, x(p, \mu')) = 0\} \text{ otherwise.} 
\end{array} \right. \]  

(25)

First, fix \( X(p) > 0 \). Consider DM’s problem at \( t=2 \) on the path \( a_1=0: \)

\[ \mu_2^* (\Omega_2, p) \in \max_{\mu} \left\{ (1 - p) \cdot \frac{E[v|\Omega_2, \mu_1^*]}{\beta} - p \cdot F - W \right\} + W, \]  

(26)
where $\Omega_2= (\emptyset, 0)$. DM’s inference is derived as follows:

$$E[v|\Omega_2 = (\emptyset, 0), \mu_1^*] = \int_{\sigma=0}^{X(p)} E[v|\sigma] \cdot f(\sigma) \cdot d\sigma$$

$$= \int_{\sigma=0}^{X(p)} \sigma^2 \cdot \exp(-\sigma) \cdot d\sigma$$

$$= 1 - \left(\frac{X(p)^2}{2} + X(p) + 1\right) \cdot \exp(-X(p))$$

$$\equiv I(X(p)),$$

where $I(\cdot)$ is monotonic in its element and has a finite codomain $(0, 1)$ over its domain $(0, \infty)$. There may exist a mixed strategy because this is a fixed point; DM uses a pure strategy if $I(X(p)) \neq Y(p)$, and a mixed strategy otherwise, as given in (21).

Next, fix $\mu_2^*(0, \emptyset; p) = \mu \in [0, 1]$. Consider DM’s problem at $t=1$. There exist a unique pure strategy almost everywhere because the value function $V_1$ is continuous in $\sigma$, and $\sigma$ is continuum. It suffices to compare two value functions as follows:

$$V_1(1, \sigma, \mu_2^*, p) - V_1(0, \sigma, \mu_2^*, p) = (1 - p) \cdot E[v|\sigma] - p \cdot F - W$$

$$+ \int_{v=0}^{\infty} \{(1 - p) \cdot v - p \cdot F - W\} \cdot f_{v|\sigma}(v|\sigma) \cdot dv$$

$$- \mu \cdot ((1 - p) \cdot E[v|\sigma] - p \cdot F - W)$$

$$= \begin{cases} 
(2 - \mu) \cdot \left(\frac{(1-p)\sigma}{2} - p \cdot F - W\right) \cdot \frac{(1-p)Y(p) - p \cdot F - W}{\sigma} \cdot Y(p) & \text{if } \sigma > Y(p) \\
(1 - \mu) \cdot \left(\frac{(1-p)\sigma}{2} - p \cdot F - W\right) & \text{otherwise.}
\end{cases}$$

Equation (28) implies:

$$V_1(1, \sigma, \mu_2^*, p) - V_1(0, \sigma, \mu_2^*, p) \begin{cases} 
> 0 \text{ for } \sigma \to \infty \\
< 0 \text{ if } \sigma < Y(p)
\end{cases}$$

and

$$\frac{\partial V_1(1, \sigma, \mu_2^*, p) - V_1(0, \sigma, \mu_2^*, p)}{\partial \sigma} \begin{cases} 
> 0 \text{ for } \sigma \in (k(p, \mu), \infty) \\
< 0 \text{ for } \sigma \in (Y(p), k(p, \mu))
\end{cases}$$

where $k(p, \mu) > Y(p)$ and:

38
\[ k(p, \mu) := \sqrt{\left( -\frac{(1-p) \cdot Y(p)}{2} + p \cdot F + W \right) \cdot \frac{2Y(p)}{(2 - \mu) \cdot (1 - p)}}. \]  

(31)

Hence, by Intermediate Value Theorem, for any \( \mu \in [0, 1] \), a unique cutoff value at \( t=1 \), denoted by \( x(p, \mu) \), is defined as follows:

\[ x(p, \mu) := \left\{ \sigma \in (Y(p), \infty) : V_1(1, \sigma, \mu_2^*, p) - V_1(0, \sigma, \mu_2^*, p) = 0, \mu_2^*(1, \sigma; p) \text{ satisfies (20), and } \mu_2^*(0, \sigma; p) = \mu \right\}. \]  

(32)

Equation (32) is simplified to (24). Finally, define a hyperplane \( h \) such that:

\[ h(p, \mu) := H(p, x(p, \mu)) = I(x(p, \mu)) - Y(p). \]  

(33)

Then, for any \( p \), a fixed point, \( X(p) \) and \( \mu_2^*(0, \sigma; p) \), is well defined as shown in (22) because \( h(p, \mu) \) and \( x(p, \mu) \) are differentiable and strictly monotone with respect to \( \mu \) such that:

\[ \frac{\partial x(p, \mu)}{\partial \mu} < 0 \text{ and } \frac{\partial h(p, \mu)}{\partial \mu} < 0 \text{ for } \forall \mu \in [0, 1]. \]  

(34)

**Proof of Proposition 2**

Fix any \( p \). Let \( \mu_2^*(0, \sigma; p) = \mu \) and \( x(p, \mu) \). The hyperplane \( h \) is expressed as follows:

\[ h(p, \mu) = I(x(p, \mu)) - Y(p) \equiv I(x) - \frac{x \cdot \beta}{1 + \sqrt{1 - \frac{(2 - \beta) \cdot \beta}{2 - \mu}}} \]  

(35)

We want to show a non-monotonic relationship between the probability of apprehension \( p \) and DM’s criminal action at \( t=2 \) on the path \( a_1 = 0 \). It suffices to show the following results:

1. For any \( \beta \), DM does not commit a crime at \( t=2 \) on the path \( a_1 = 0 \) at the limit \( p \rightarrow 1 \):

\[ \lim_{p \rightarrow 1} \mu_2^*(0, \sigma; p) = 0 \Leftrightarrow \lim_{p \rightarrow 1} h(p, 0) < 0. \]  

(36)

2. There exist \( \beta \in (B_1, B_2) \) such that DM does not commit a crime at \( t=2 \) on the path \( a_1 = 0 \) at the limit \( p \searrow 0 \):

\[ \lim_{p \searrow 0} \mu_2^*(0, \sigma; p) = 0 \Leftrightarrow \lim_{p \searrow 0} h(p, 0) < 0, \]  

(37)
and DM does not commit a crime at \( t=2 \) on the path \( a_1=0 \) at some \( p \):

\[
\max_{p \in (0,1)} \mu_2^* (0, \varnothing; p) = 1 \Leftrightarrow \max_{p \in (0,1)} h(p, 1) > 0. \tag{38}
\]

(36) holds because DM’s inference is bounded \( I(x) < 1 \), but \( \lim_{p \downarrow 1} Y(p) = \infty \). There is a lower bound \( B_1 \) so that (37) holds if and only if \( \beta > B_1 \) holds where:

\[
B_1 := \left\{ \beta \in (0, 1) : \lim_{p \downarrow 0} h(p, 0) = I(x) - \beta \cdot W = 0 \right. \\
\left. \text{and } x = \lim_{p \downarrow 0} x(p, 0) = W \cdot \left( 1 + \sqrt{1 + \frac{(1-\beta)^2}{2}} \right) \right\}. \tag{39}
\]

This is true because \( \frac{\partial h(p, \mu)}{\partial \beta} < 0 \) over \( \beta \in (0, 1) \), \( \lim_{\beta \downarrow 0, p \downarrow 0} I(x) = \infty \), \( \lim_{\beta \downarrow 1, p \downarrow 0} h(p, 0) < 0 \).

There is an upper bound \( B_2 \) such that (38) holds for \( \beta < B_2 \), where:

\[
B_2 := \left\{ \beta \in (0, 1) : \frac{\partial l(x, \beta)}{\partial x} \left( \frac{x^2}{2} + x + 1 \right) \cdot \exp(-x) - \frac{x \cdot \beta}{2 - \beta} = 0 \right. \\
\left. \text{and } x = \lim_{p \downarrow 0} x(p, 0) = W \cdot \frac{\beta}{2 - \beta} \right\}. \tag{40}
\]

This upper bound is derived as follows. Let \( x=x(p, 1) \). Then, define \( l(x; \beta) \) such that:

\[
l(x; \beta) := 1 - \left( \frac{x^2}{2} + x + 1 \right) \cdot \exp(-x) - \frac{x \cdot \beta}{2 - \beta}. \tag{41}
\]

\( l(x, \beta) \) is convex for \( x < 2 \) and concave for \( x > 2 \); \( \frac{\partial l(x, \beta)}{\partial x} \) is maximized at \( x = 2 \). Recall \( h(p, 1) = I(x) - Y(p) \equiv l(x; \beta) \).

It suffices to find a range of \( \beta \) given which there exists \( x > 2 \) satisfying:

\[
l(x, \beta) = 1 - \left( \frac{x^2}{2} + x + 1 \right) \cdot \exp(-x) - \frac{x \cdot \beta}{2 - \beta} > 0, \tag{42}
\]

\[
\frac{\partial l(x, \beta)}{\partial x} = \frac{x^2}{2} \cdot \exp(-x) - \frac{\beta}{2 - \beta} = 0. \tag{43}
\]

(42) and (43) are equivalent to \( \beta < B_2 \), where \( B_2 \) is given by (40). Consequently, if \( \beta \in (B_1, B_2) \), there exist cutoffs \( 0 < P_1 < P_3 < P_4 < P_2 < 1 \), where:

\[
h(P_1, 0) = h(P_2, 0) = 0 \text{ and } h(P_3, 1) = h(P_4, 1) = 0. \tag{44}
\]
Hence, a non-monotonic relation between \( p \) and \( \mu_2^* (0, \varnothing; p) \) is observed as follows:

\[
\mu_2^* (0, \varnothing; p) \in \begin{cases} 
\{0\} & \text{if } p \in (0, P_1) \\
(0, 1) & \text{if } p \in (P_1, P_3) \\
\{1\} & \text{if } p \in (P_3, P_4) \\
(0, 1) & \text{if } p \in (P_4, P_2) \\
\{0\} & \text{if } p \in (P_2, 1). 
\end{cases}
\]  

\( \mu_2^* (0, \varnothing; p) \) is increasing in \( p \) for \( p \in (P_1, P_3) \) while it is decreasing in \( p \) otherwise.

For example, given \( \beta=0.3, F=0.5 \) and \( W=0.4 \), the cutoffs are \( B_1 \approx 0.11, B_2 \approx 0.32, P_1 \approx 0.25, P_2 \approx 0.55, P_3 \approx 0.33 \) and \( P_4 \approx 0.52 \). As a result, non-monotonic relationship is observed as shown in Figure 3, 4 and 5.

### 8.1 Proof of Proposition 3

Fix any \( p \). Let \( \mu_2^* (0, \varnothing; p) = \mu \) and \( x=x(p, \mu) \). The total number of criminal actions are derived as follows:

\[
E [a_1 + a_2] = \int_0^\infty \int E [\mu_1^* (\sigma) + \mu_2^* (\Omega_2) |\sigma] \cdot f_\sigma (\sigma) \cdot d\sigma
\]

\[
= \int_{\sigma=x}^{\infty} 2 \cdot f_\sigma (\sigma) \cdot d\sigma + x \cdot \int_{\sigma=0}^{\infty} f_\sigma (\sigma) \cdot d\sigma = \mu + (2 - \mu) \cdot (x + 1) \cdot \exp (-x).
\]

Given \( \mu=1 \):

\[
E [a_1 + a_2] = 1 + (x + 1) \cdot \exp (-x), \tag{47}
\]

where \( x = x(p, 1) = \left( \frac{p \cdot F + W}{1 - p} \right) \cdot (2 - \beta) \).

Given \( \mu=0 \):

\[
E [a_1 + a_2] = 2 \cdot (x + 1) \cdot \exp (-x), \tag{48}
\]

where \( x = x(p, 1) = \left( \frac{p \cdot F + W}{1 - p} \right) \cdot \left( 1 + \sqrt{\frac{1 + (1 - \beta)^2}{2}} \right) \).
In either case, $E[a_1 + a_2]$ is decreasing in $p$. Hence, it suffices to show the following inequality:

$$2(x + 1) \cdot \exp(-x) < 1 + (x' + 1) \cdot \exp(-x'), \quad (49)$$

where:

$$x : = x(P_1, 0) = \left( \frac{P_1 \cdot F + W}{1 - P_1} \right) \cdot \left( 1 + \sqrt{\frac{1 + (1 - \beta)^2}{2}} \right),$$

$$x' : = x(P_3, 1) = \left( \frac{P_3 \cdot F + W}{1 - P_3} \right) \cdot (2 - \beta),$$

$$P_1 : \in \min \{ p \in (0, 1) : h(p, 0) = 0 \},$$

$$P_3 : \in \min \{ p \in (0, 1) : h(p, 1) = 0 \}.$$

(49) holds if $x' / x = 1$. It may not hold for very large $x'$ / $x$. And $x' / x$ increases as $\frac{h(p, 0)}{h(p, 1)}$ increases; $\frac{h(p, 0)}{h(p, 1)}$ is increasing in $\beta$.

Hence, it suffices to find the upper bound for $\beta$ such that (49) holds. First suppose $\beta = B_2$, the upper bound for Proposition 2, which implies $\frac{x^2}{2} \cdot \exp(-x) = \frac{\beta}{2 - \beta}$.

Hence, for $\beta = B_2$, $x = x(P_1, 0)$ and $x' = x(P_2, 1)$ satisfy:

$$x \in \left\{ t : h(P_1, 0) = 0 \text{ and } \frac{t^2}{2} \cdot \exp(-t) = \frac{\beta}{2 - \beta} \right\} \quad (50)$$

$$\Rightarrow 2(x + 1) \cdot \exp(-x) = 2 - \frac{2\beta}{2 - \beta} - \frac{2x \cdot \beta}{1 + \sqrt{\frac{1 + (1 - \beta)^2}{2}}},$$

and

$$x' \in \left\{ t : h(P_3, 1) = 0 \text{ and } \frac{x'^2}{2} \cdot \exp(-x') = \frac{\beta}{2 - \beta} \right\} \quad (51)$$

$$\Rightarrow 1 + (x' + 1) \cdot \exp(-x') = 2 - \frac{\beta}{2 - \beta} - \frac{x' \cdot \beta}{2 - \beta}.$$

(50) and (51) imply (49) because:

$$2(x + 1) \cdot \exp(-x) < 1 + (x' + 1) \cdot \exp(-x') \quad (52)$$

$$\Leftrightarrow \frac{x' - 1}{x} < \frac{2(2 - \beta)}{1 + \sqrt{\frac{1 + (1 - \beta)^2}{2}}}.$$
The last inequality holds because:

\[
x' - 1 = \frac{2 - \beta - \frac{1-\beta}{pF+W}}{1 + \sqrt{\frac{1+(1-\beta)^2}{2}}} < \frac{2 - \beta - \frac{1}{W}}{1 + \sqrt{\frac{1+(1-\beta)^2}{2}}} < \frac{2(2 - \beta)}{1 + \sqrt{\frac{1+(1-\beta)^2}{2}}}.
\]  

(53)

Hence, \( \beta = B_2 \) is the upper bound such that (49) holds. Hence, (49) holds for \( \beta \in (B_1, B_2) \).

For example, given \( \beta = 0.3, F = 0.5 \) and \( W = 0.4 \), non-monotonic relationship is observed as shown in Figure 6.