Diana Barro, Elio Canestrelli

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Portfolio management with minimum guarantees: some modeling and optimization issues

Diana Barro  
<d.barro@unive.it>  
Dept. of Applied Mathematics  
University of Venice

Elio Canestrelli  
<canestre@unive.it>  
Dept. of Applied Mathematics  
University of Venice

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Abstract. In this contribution we consider a dynamic portfolio optimization problem where the manager has to deal with the presence of minimum guarantee requirements on the performance of the portfolio. We briefly discuss different possibilities for the formulation of the problem and present a quite general formulation which includes transaction costs, cardinality constraints and buy-in thresholds. The presence of realistic and operational constraints introduces binary and integer variables greatly increasing the complexity of the problem.

Keywords: Minimum guarantee, dynamic portfolio management, scenario.

JEL Classification Numbers: C61, G11.


Correspondence to:
Diana Barro  
Dept. of Applied Mathematics, University of Venice  
Dorsoduro 3825/e  
30123 Venezia, Italy  
Phone: [+39] (041)-234-6923  
Fax: [+39] (041)-522-1756  
E-mail: d.barro@unive.it
1 Introduction

Interest in financial products with minimum guarantee features has increased in response to a period of financial market instability and low interest rate levels. Different categories of investors are looking for risk-return profiles which are capable of integrating potentiality for upward capture ratios and more effective mechanisms of control of the risk of downward movements.

A possible way of achieving these goals is to introduce low barriers to shape the risk-return profile. According to the chosen level, these barriers can be expressed as minimum return requirements or maximum amount of loss allowed.

There are many different types of products and policies which offer risk-return profiles with minimum guarantee features. In this paper we are interested in optimal portfolio strategies which could be adopted by the manager of a fund linked with the issuance of these products.

Policies which give a minimum guaranteed return usually provide also a certain amount of the return of the risky part of the portfolio invested in the equity market. Limiting the portfolio allocation only to bonds and liquidity instruments do not allow upside potential to be captured. The main objective is thus to combine a guaranteed return, i.e. a low profile of risk, and the possibility of achieving, at least, a part of the higher returns which could be granted by the equity market at the cost of a high exposure to the risk of not meeting the minimum return requirement.

Different contributions in the literature tackled the problem of optimal portfolio choices with the presence of a minimum guarantee. For example, we refer to [10] and [11], for a discussion of these problems in a continuous time framework. Another interesting issue is related to the pricing problem for products including minimum guarantee features, see, for example, [2],[3].

We consider the problem of formulating and solving an optimal allocation problem including a minimum guarantee requirement and a participation in the returns generated from the risky portfolio. These goals can be achieved both considering them as constraints or including them in the objective function. We discuss the problem in a dynamic optimization framework in presence of frictions in the market. We formulate the problem as a multistage stochastic programming problem in arborescent formulation. The presence of elements such as binary variables and roundlots constraints do not allow us to rely on efficient decomposition techniques already used for other dynamic optimization problems, see, for example, [4]. Thus to tackle the resulting problem we are interested in the use of artificial intelligence approaches. Different contributions in the literature pointed out that these techniques can be efficiently applied to solve a variety of portfolio optimization problems, see, for example, [20],[22],[25].

2 Minimum guarantee requirements

In this contribution we focus on a dynamic portfolio management problem in which we consider an investor who is interested in maximizing his wealth controlling in the meantime
the downside risk. Different approaches have been proposed in the literature both in discrete and continuous time using appropriate measure of downside risk or proper constraints on the level of shortfall allowed. In this contribution we want to focus on models where the maximization of wealth is coupled with the presence of a minimum guarantee return for the portfolio. The control of risk is set fixing a lower bound to the level of wealth which can be reached by the portfolio rather than controlling the variability of the distribution of returns of the portfolio. This approach seems to be interesting and appealing in different financial context where the investors are risk averse and may require guarantees to be convinced to enter the investment. We can find many example of this type of risk control profile in financial products like pension funds, life insurance contract, unit linked policies.

We are not interested in the pricing of these products, we are rather interested in the point of view of a manager who has to manage a fund linked with these products and thus our goal is to study the formulation and the solution of a dynamic optimization problem.

Different strategies can be implemented to build the desired risk/return payoff. In particular we can include direct hedging using derivatives, or indirect hedging the long positions using short-selling. Other strategies can be built through a re-balancing of the portfolio changing the proportions invested in a risky fund and the riskless component, but without an optimization in the composition of the risky fund. A simple example in a static framework is the stop loss strategy while in a dynamic context we can built strategies such as the constant proportion portfolio insurance. In this contribution we are more interested in analyzing a dynamic optimization problem in which we deal with the requirement of a minimum guarantee and we are interested in devising an effective way of introducing it in the formulation of the problem analyzing the effect of different choices on the usable solution approaches.

In the following we consider two different ways of including the minimum guarantee requirements in the formulation of the problem. Firstly the condition can be included in the objective function as the goal, or one of the goals, of the optimization problem. Secondly it can be handled through the introduction of one or more constraints to the optimization problem. In the latter case we can consider both hard constraints or probabilistic constraints which allows to reach the goal within a specified level of probability.

In the following we consider the first approach and include the goal of minimizing the shortfall with respect to the minimum guarantee in the objective function. We consider a portfolio made of three asset classes, i.e. equities, bonds and cash. In this contribution we do not consider hedging strategies, neither direct hedging, including derivatives in the portfolio, nor indirect ones through short selling. The presence of these elements could help in the achievements of the desired goals but their inclusion in the portfolio require a careful modeling of operational constraints linked with these positions, such as borrowing limits and the adequacy of margins which need to be monitored.

We consider a dynamic optimization problem in discrete time with a finite horizon and a scenario framework for the description of the uncertainty included in the problem. In the following $g_{k_t}, t = 1, \ldots, n_1$, denotes the position in the $i$-th stock and $b_{j_k}, j = 1, \ldots, n_2$, denotes the position in the $j$-th bond, respectively, while $c_{k_t}$ denotes the amount of cash.

We denote with $r_{k_t} = (r_{1_k}, \ldots, r_{n_k})$ the vector of returns of the risky assets for the period $[t - 1, t]$ in node $k_t$ and with $r_{c_{k_t}}$ the return on the liquidity component in node
In order to account for transaction costs and liquidity component in the portfolio we introduce two vector of variables $a_k = (a_{1k}, \ldots, a_{nk})$ and $v_k = (v_{1k}, \ldots, v_{nk})$ denoting the value of each asset purchased and sold at time $t$ in node $k_t$, while we denote with $\kappa^+$ and $\kappa^-$ the proportional transaction costs for purchases and sells.

We denote with $\psi_k(y_k, z_t)$ a generic distance measure between the value of the portfolio and the value of the minimum guarantee in node $k_t$ at time $t$.

We propose the absolute downside deviation as measure of distance between the managed portfolio $y_k$ and the minimum guarantee benchmark $z_t$

$$\psi_k(y_k, z_t) = [y_k - z_t]^- = -\min[y_k - z_t, 0] = \gamma_k^-.$$  

If we consider the minimization of the mean absolute downside deviation for the terminal period, the objective function becomes

$$\min \frac{1}{S} \sum_{s=1}^{S} \pi_s \gamma_s^- \quad (2)$$

where $s$ denotes a scenario, i.e. a path connecting the origin of the tree to a leaf node, and $\pi_s$ denotes the probability associated to the path. This choice results in a linear optimization problem which can be solved quite efficiently even if the number of scenarios included is high.

Other choices are possible for the minimum guarantee goal, in particular, for example, we can minimize the maximum downside deviation at the end of the horizon. To this aim we define a non negative variable $\theta_s$ as the upper bound of the absolute deviations, i.e. $\theta_s \geq |y_s - z_t|$; this problem, too, can be transformed in a linear optimization problem, see [23], obtaining

$$\min \theta_s$$
$$\theta_s + y_s \geq z_T \quad s = 1, \ldots, S \quad (3)$$

The minimum return goal in both the forms described above can be combined with a measure of performance such as the maximization of the wealth in each period, in such a way that a first part of the objective function is related to the maximization of the value of the wealth in each period (an expected utility from wealth can be introduced to take into account risk averse behavior), while a second part controls the downside risk minimizing the mean absolute downside deviation or the maximum downside deviation from the guarantee return portfolio. Alternatively we can set a minimum guarantee return in each period. This formulation would allow us to introduce a path dependent goal in which we can lock in part of the return of the portfolio in such a way that the level of the minimum guarantee can be increased during the management period in case of positive returns in the portfolio, as it often is, for example, for some financial products with the introduction of lock-in conditions.

The minimum guarantee can be assumed constant over the entire planning horizon or it can follow a deterministic dynamics, i.e it is not scenario dependent.
A second approach in dealing with the minimum guarantee requirements is based on the introduction of some constraints in the formulation of the optimization problem to force the level of wealth or the return of the portfolio to satisfy the required lower bound. The constraints can be of different types, in particular we can consider both hard constraints and probabilistic constraints. In the first case we can introduce constraints on the level of wealth or on the return of the portfolio in the form of lower bounds while in the second we can require that the goal is reached with a specified level of confidence.

3 Formulation of the problem

A crucial aspect for the formulation of these problems is the definition of a set of operational constraints which are particularly relevant in the case of direct or indirect hedging using derivatives or short-selling since they are related to the presence of margins and borrowing limits. Nevertheless they are particularly interesting in the management of funds with minimum guarantee requirements since very often they are subject to operational constraints. In particular we are interested in focusing on the constraints which increase the computational complexity of the problem, i.e. discrete restrictions which results in the introduction of binary or integer variables. Different classes of constraints have been considered in the literature, mainly in static portfolio problems, see, for example [17] and references therein.

Following [17] we consider constraints to explicitly limit the number of stocks which can be included in the portfolio, i.e. cardinality constraints; buy-in thresholds which define the minimum level below which the asset is not included; roundlots constraints which are defined as the discrete numbers of assets which are taken as the basic unit of investment.

Given a set of binary variables \( \delta_{it} \), \( \delta_{it} \in \{0, 1\} \), \( i = 1, \ldots, N \), such that \( \delta_{it} \) is equal to zero if the asset \( i \) is not included in the portfolio at time \( t \); the cardinality constraints can be expressed as

\[
\sum_{i=1}^{N} \delta_{it} = N_{t}^{max}
\]

where \( N_{t}^{max} \) represents the number of assets which can be included in the portfolio. The presence of cardinality constraints is linked with buy-in thresholds. Using the binary variables introduced jointly with upper and lower bounds \( l_{it} \) and \( u_{it} \) for asset \( i \) at time \( t \), we can express the existence of buy-in thresholds as

\[
l_{it}\delta_{it} \leq x_{it} \leq u_{it}\delta_{it}
\]

where \( x_{it} \) represents the fraction of portfolio wealth invested in the generic asset \( i \) at time \( t \), and is defined as \( q_{ik}/y_{ik} \) for equities, and \( b_{ik}/y_{ik} \) for bonds.

In the following we will present the formulation of a multistage stochastic programming problem, in its arborescent formulation, with the introduction of constraints on the presence of minimum guaranteed return for the portfolio for each period of the investment horizon and the operational constraints.
\[
\begin{align*}
\min_{q_t, b_t, c_t} & \quad \sum_{t=1}^{T} \left[ \alpha_t \sum_{k_t=K_{t-1}+1}^{K_t} \psi(y_{k_t}) - \beta_t \sum_{k_t=K_{t-1}+1}^{K_t} \phi(y_{k_t}) \right] \\
\text{s.t.} & \quad y_{k_t} = c_{k_t} + \sum_{i=1}^{n_1} q_{i,k_t} + \sum_{j=1}^{n_2} b_{j,k_t} \\
&q_{i,k_t} = (1 + r_{i,k_t}) \left[ q_{i,f(k_t)} + a_i f(k_t) - v_{i,f(k_t)} \right] \quad i = 1, \ldots, n_1 \\
b_{j,k_t} = (1 + r_{j,k_t}) \left[ b_j f(k_t) + a_j f(k_t) - v_{j,f(k_t)} \right] \quad j = 1, \ldots, n_2 \\
c_{k_t} = (1 + r_{c,k_t}) [c_f(k_t) - \sum_{i=1}^{n_1} (\kappa^+) a_i f(k_t) + \sum_{i=1}^{n_1} (\kappa^-) v_i f(k_t)] \\
& \quad + \sum_{j=1}^{n_2} (\kappa^+) a_j f(k_t) + \sum_{j=1}^{n_2} (\kappa^-) v_j f(k_t) + \sum_{j=1}^{n_2} g_{k_t} b_{j,f(k_t)} \\
& \quad a_{i,k_t} \geq 0 \quad v_{i,k_t} \geq 0 \quad i = 1, \ldots, n_1 \\
a_{j,k_t} \geq 0 \quad v_{j,k_t} \geq 0 \quad j = 1, \ldots, n_2 \\
q_{i,0} = \bar{q}_i \quad i = 1, \ldots, n_1 \\
b_{j,0} = \bar{b}_j \quad j = 1, \ldots, n_2 \\
c_0 = \bar{c} \\
y_{k_t} \geq z_t \\
\delta_{i,k_t} \in \{0, 1\} \quad i = 1, \ldots, n \\
\sum_{i=1}^{n} \delta_{i,k_t} = N^{max} \\
l_{i,k_t} \delta_{i,k_t} \leq x_{i,k_t} \leq u_{i,k_t} \delta_{i,k_t} \\
k_t = K_{t-1} + 1, \ldots, K_t \\
t = 1, \ldots, T
\end{align*}
\]

where \( \psi(y_{k_t}) \) and \( \phi(y_{k_t}) \) denotes two functions of the level of wealth in the portfolio accounting for the return and the risk of the portfolio, respectively. The choice of these function characterizes the resulting optimization problem, in particular the manager may be interested in maximizing the expected return of the portfolio or the expected utility of wealth and at the same time minimize the downside deviations or the variance of the portfolio. A proper choice of these functions may lead to a linear multistage stochastic programming problem. Equations (5)-(8) represent the portfolio composition in node \( k_t \), the dynamics of the amounts of stocks, bonds and cash in the portfolio moving from the ancestor node \( f(k_t) \), at time \( t - 1 \), to the descendent nodes \( k_t \), at time \( t \), with \( K_0 = 0 \), respectively. With \( g_{k_t} \) in equation (8) we denote the inflows from the bonds in the portfolio.
Following [12] we assume that there is an annual guaranteed rate of return denoted with $\rho$. If the initial wealth is $W_0 = \sum_{i=1}^{n+1} x_i 0$, then the value of the guarantee at the end of the planning horizon is $W_T = W_0 (1 + \rho)^T$. At each intermediate date the value of the guarantee is given by $z_t = e^{\delta(t,T-t)}(T-t) W_0 (1 + \rho)^T$, where $e^{\delta(t,T-t)}(T-t)$ is a discounting factor, i.e. the price at time $t$ of a zcb which pays 1 at terminal time $T$.

If we omit the operational constraints we obtain a (linear) multistage stochastic programming problem which can be efficiently solved using decomposition techniques, see for example [4] and references therein.

The presence of discrete constraints greatly increases the computational complexity of the resulting problem, see, for example, [17] for a discussion of the effects of the introduction of these constraints in a static portfolio selection problem. Different computational approaches based on heuristic algorithms (like genetic algorithms, neural networks, simulated annealing, etc.), have been proposed to deal with this kind of problems.

In particular several papers report successful results from the application of artificial intelligence tools to tackle complex financial problems, see for example [20][22][19].

We think this promising approach could help in tackling the family of problems we are interested in. In particular we are interested in comparing the portfolio strategies obtained using different objective functions.

A first issue we must deal with is related to the choice of how these techniques can be applied to solve multistage stochastic optimization problems. In particular we suggest to apply them directly to solve the deterministic equivalent problem which is a large scale optimization problem.

4 Concluding remarks

In this contribution we analyze the issue of managing a portfolio with the goal of building a profile of risk/return characterized by the presence of a minimum guarantee. We would like to focus on different modeling choices which are available and on the complexity of the resulting optimization problems. In this work we do not consider hedging strategies obtained using derivatives or short selling, this topic is left for future research, as well as a comparison of the effectiveness of the different strategies in reaching the goal of a minimum guarantee return portfolio comparing them also from the point of view of the costs associated with the dynamic management of the portfolio.

References


