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Some proposals about multivariate risk measurement

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Abstract. In actuarial literature the properties of risk measures or insurance premium principles have been extensively studied. In our work we propose a characterization of some particular classes of multivariate and bivariate risk measures. Given two random variables we can define an univariate integral stochastic ordering by considering a set of functions, that, through their peculiar properties, originate different stochastic orderings. These stochastic order relations of integral form may be extended to cover also the case of random vectors. It is, in fact, proposed a kind of stop-loss premium, and then a stop-loss order in the multivariate setting and some equivalent conditions. We propose an axiomatic approach based on a minimal set of properties which characterizes an insurance premium principle. In the univariate case we know that Conditional Value at Risk can be represented through distortion risk measures and a distortion risk measure can be viewed as a combination of CVaRs, we propose a generalization of this result in a multivariate framework. In the bivariate case we want to compare the concept of risk measure to that one of concordance measure when the marginals are given.

Keywords: Risk measures, distortion function, concordance order, concordance measure.

JEL Classification Numbers: D810.


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1 Introduction

In Actuarial Sciences it is almost usual to compare two random variables, that are risks, by stochastic orderings defined through inequalities on expectations of the random variables transformed by measurable functions. More precisely, given two random variables $X$ and $Y$, an univariate integral stochastic ordering $\preceq_F$ is defined as follows

$$ X \preceq_F Y \iff E[\phi(X)] \leq E[\phi(Y)] $$

for all $\phi \in F = \{ \phi : \mathbb{R}_+ \to \mathbb{R}_+ \}$, for which the expectations exist. By characterizing the considered set of functions some particular stochastic orderings may be obtained such as stochastic dominance or stop-loss order. These stochastic order relations of integral form may be extended to cover also the case of random vectors. In particular in [3] the bivariate extension of a class of univariate orderings of convex-type is considered.

In this paper we propose an axiomatic approach based on a minimal set of properties which characterize an insurance premium principle; in particular we study the extension to the multivariate case of distorted risk measures and we propose a new kind of vector risk measure.

2 Multivariate case

We consider only non-negative random vectors. Let $\Omega$ be the space of the states of nature, $\mathcal{F}$ be the $\sigma$-field and $P$ be the probability measure on $\mathcal{F}$. Our random vector is the function $X : \Omega \to \mathbb{R}^n_+$ and such that $X(\omega)$ represents the payoff obtained if state $\omega$ occurs.

We also specify some notations: $F_X(x) : \mathbb{R}^n \to [0,1]$ is the distribution function of $X$, $S_X(x) : \mathbb{R}^n \to [0,1]$ is its survival or tail function, and $(X(\omega) - a)_+ = \max(X(\omega) - a, 0)$ componentwise.

A risk measure, or a premium principle, is the functional $R : \mathcal{X} \to \tilde{\mathbb{R}}$, where $\mathcal{X}$ is a set of non-negative random vectors and $\tilde{\mathbb{R}}$ is the extended real line.

Some desirable properties for risk measures are:

1. **Expectation boundedness** or non-negative loading: $R[X] \geq E[X_1 \ldots X_n] \forall X$;

2. **Non-excessive loading**: $R[X] \leq \sup_{\omega \in \Omega} \{|X_1(\omega)|, \ldots, |X_n(\omega)|\}$;

3. **Translation invariance**: $R[X + a] = R[X] + \bar{a} \forall X, \forall a \in \mathbb{R}^n$, where $a$ is a vector of sure initial amounts and $\bar{a}$ is the componentwise product of the elements of $a$;

4. **Positive homogeneity of order n**: $R[cX] = c^n R[X] \forall X, \forall c \geq 0$;

5. **Monotonicity**: $R[X] \leq R[Y] \forall X, Y$ such that $X \preceq Y$ in some stochastic sense;

6. **Subadditivity**: $R[X + Y] \leq R[X] + R[Y] \forall X, Y$ that reflects the idea that risk can be reduced by diversification;

7. **Constancy**: $R[b] = \bar{b} \forall b \in \mathbb{R}^n$. A special case is $R[0] = 0$, that is called normalization property;
8. **Convexity:** \( R[\lambda X + (1 - \lambda) Y] \leq \lambda R[X] + (1 - \lambda) R[Y] \), \( \forall X, Y \) and \( \lambda \in [0, 1] \); this property implies diversification effects as subadditivity does.

We also recall some notions about stochastic orderings for multivariate random variables. The concept of stochastic dominance is fundamental to compare two risky portfolios, but doesn’t allow us to determine a complete order among all the risky investments. The usual multivariate stochastic order is defined as follows:

**Definition 2.1** Let \( X \) and \( Y \) be two random vectors that take on values in \( \mathbb{R}^n_+ \); we say that \( X \) is smaller than \( Y \) in the sense of usual stochastic dominance, \( X \preceq_{SD} Y \), if

\[
E[\phi(X)] \leq E[\phi(Y)] \quad \forall \phi : \mathbb{R}^n \to \mathbb{R}
\]

is a non-decreasing function for which the expectation exists.

Now we introduce the concept of supermodular functions that play a central role in modeling concordance among random vectors.

We endowed \( \mathbb{R}^n \) with the usual product order. With this order \( \mathbb{R}^n \) becomes a lattice and we denote the supremum and the infimum of \( x \) and \( y \) by \( x \lor y \) and \( x \land y \) respectively.

**Definition 2.2** A function \( f : D \subseteq \mathbb{R}^n \to \mathbb{R} \) is supermodular if

\[
f(x \lor y) + f(x \land y) \geq f(x) + f(y)
\]

when \( x, y, x \lor y, x \land y \in D \).

Supermodularity may also be defined in terms of increasing differences and so the following proposition characterizes supermodular functions.

**Proposition 2.1** Let \( f \) be a function \( D \subseteq \mathbb{R}^n \to \mathbb{R} \). \( f \) is supermodular if and only if when \( x, y \in D \)

\[
f(x + \mathbf{a} + \mathbf{b}) - f(x + \mathbf{b}) \geq f(x + \mathbf{a}) - f(x)
\]

for all \( \mathbf{a}, \mathbf{b} \) with \( \mathbf{a}, \mathbf{b} \geq 0 \), such that \( x + \mathbf{a}, x + \mathbf{b}, x + \mathbf{a} + \mathbf{b} \in D \).

Increasing difference transfers the supermodularity condition to one involving the linear structure of \( \mathbb{R}^n \). It is worth noting that supermodularity condition is only an “inter-attribute” relation. Intuitively increasing differences say that there must be “complementarity” among attributes. So a supermodular function has the interpretation of a “complementarity” condition of the inputs to be aggregated and a tendency of a collection of high scores to reinforce each other. Clearly, indicator functions associated to upper orthants or lower orthants, denoted by \( f = 1_{[t, \infty)} \) and \( f = 1_{(-\infty, t]} \) respectively, where \( [t, +\infty) = \{ x : x \in \mathbb{R}^2 \land x_1 \geq t_1, x_2 \geq t_2 \} \) and \( (-\infty, t] = \{ x : x \in \mathbb{R}^2 \land x_1 \leq t_1, x_2 \leq t_2 \} \) for a given \( t = (t_1, t_2) \in \mathbb{R}^2 \), are supermodular.

There are many interesting representations of supermodular functions and they are used in many different applications, we present the supermodular order (SM), that can be viewed as a multivariate extension of concordance order (see bivariate case next).
Definition 2.3 Let $X$ and $Y$ be two vectors that take on values in $\mathbb{R}^n$ such that $E[f(X)] \leq E[f(Y)]$ for all supermodular functions $f : \mathbb{R}^n \to \mathbb{R}$, provided the expectation exists. Then $X$ is said to be smaller than $Y$ in the supermodular order ($X \preceq_{SM} Y$).

Two other possible generalizations of the univariate stochastic order are the orthant orders, based on the no-crossing condition between the respective distribution functions or the tail functions. Whereas we had equivalence in the univariate case, these become distinct in the multivariate setting.

Definition 2.4 If it holds
$$S^X(x) \leq S^Y(x) \quad \forall x \in \mathbb{R}^n,$$
then $X$ is said to be smaller than $Y$ in the upper orthant order, denoted by $X \preceq_{UO} Y$.

Definition 2.5 If it holds
$$F^X(x) \geq F^Y(x) \quad \forall x \in \mathbb{R}^n,$$
then $X$ is said to be smaller than $Y$ in the lower orthant order, denoted by $X \preceq_{LO} Y$.

This terminology raises from the fact that the sets $\{x \in \mathbb{R}^n : x > a\}$, for fixed $a$, are called upper orthants, while the sets $\{x \in \mathbb{R}^n : x \leq a\}$ are called lower orthants.

When $Y \preceq_{LO} X$ and $X \preceq_{UO} Y$ simultaneously hold, $X$ is said to be smaller than $Y$ in the concordance order.

From the definitions of orders presented above we can immediately deduced the following implications:

$$X \preceq_{SM} Y \implies X \preceq_{UO} Y$$
$$X \preceq_{SM} Y \implies Y \preceq_{LO} X.$$

Let us now to characterize another formulation for stop-loss transform in the multivariate setting.

Definition 2.6 The product stop-loss transform of a random vector $X \in \mathcal{X}$ is defined by:
$$\pi^X(t) = E[(X_1 - t_1)_+ \ldots (X_n - t_n)_+] \quad \forall t \in \mathbb{R}^n.$$  

As in the univariate case, we can use this instrument to derive a multivariate stochastic order:

Definition 2.7 Let $X, Y \in \mathcal{X}$ be two random vectors. We say that $X$ preceeds $Y$ in the multivariate product stop-loss order ($X \preceq_{SL} Y$) if it holds:
$$\pi^X(t) \leq \pi^Y(t) \quad \forall t \in \mathbb{R}^n.$$

If $n = 2$, obviously $\forall t = (t_1, t_2) \in \mathbb{R}^2$ we have:
$$X \preceq_{SL_2} Y \iff E[(X_1 - t_1)_+(X_2 - t_2)_+] \leq E[(Y_1 - t_1)_+(Y_2 - t_2)_+].$$
2.1 Bivariate orders

It could be interesting give some extensions to the theory of risk in the multivariate case, but sometimes it is not possible and we will be satisfied if the generalization will work at least in two dimensions. In many situations individual risks are correlated since they are subject to the same claim generating mechanism or are determined by the same environment. However, in traditional risk theory, individual risks are usually assumed to be independent for tractability. In recent years the study of the impact of dependence among risks has become a major topic, in particular in Actuarial Sciences. Several notions of dependence were introduced to model the fact that larger values of one of the component of a multivariate risk tend to be associated with larger values of the others. As it is well-known, different notions are equivalent in the bivariate case for risks with the same univariate marginal distribution ([12]) but this is no longer true for \( n \)-variate risks with \( n \geq 3 \) ([8]).

We introduce now the concept of Fréchet space: \( \mathcal{R} \) denotes the Fréchet space given the margins, that is \( \mathcal{R}(F_1, F_2) \) is the class of all bivariate distributions with given margins \( F_1, F_2 \).

The lower Fréchet bound \( X \) of \( X \) is defined by

\[
F^X(t_1, t_2) := \max\{F_1(t_1) + F_2(t_2) - 1, 0\}
\]

and the upper Fréchet bound of \( X \), \( \bar{X} \), is defined by

\[
F^{\bar{X}}(t) := \min_i \{F_i(t_i)\},
\]

where \( t = (t_1, t_2) \in \mathbb{R}^2 \) and \( i = 1, 2 \).

**Definition 2.8** Consider two random vectors \( X, Y \in \mathcal{R}(F_1, F_2) \). If

\[
F^X(t_1, t_2) \leq F^Y(t_1, t_2) \ \forall t \in \mathbb{R}^2,
\]

or equivalently, if

\[
S^X(t_1, t_2) \leq S^Y(t_1, t_2) \ \forall t \in \mathbb{R}^2,
\]

then \( X \) is said to be smaller than \( Y \) in the concordance order \( X \preceq_C Y \).

The equivalence holds because it follows from the relation:

\[
F^X(x_1, x_2) = 1 - S_1(x_1) - S_2(x_2) + S^X(x_1, x_2).
\]

**Theorem 1** Let \( X, Y \) be bivariate random variables, where \( X, Y \in \mathcal{R}(F_1, F_2) \). Then

\[
X \preceq_{UO} Y \iff Y \preceq_{LO} X \iff X \preceq_{SM} Y \iff X \preceq_{C} Y.
\]

This result is no longer true when multivariate random variables are considered with \( n \geq 3 \).

Moreover the following result holds:

**Theorem 2** Let \( X, Y \) be bivariate random variables in \( \mathcal{R}(F_1, F_2) \). The following conditions are equivalent:

i) \( X \preceq_{SM} Y \);

ii) \( E[f(X)] \leq E[f(Y)] \) for every increasing supermodular function \( f \);

iii) \( E[f_1(X_1)f_2(X_2)] \leq E[f_1(Y_1)f_2(Y_2)] \) for all increasing functions \( f_1, f_2 \);
iv) $\pi_X(t) \leq \pi_Y(t)$ $\forall t \in \mathbb{R}^2$.

Proof: Condition i) obviously implies ii). Conversely, let condition ii) be true. Since every supermodular function is limit of increasing supermodular functions, then (see Theorem 3.4 in [7]) the validity of condition i) is ensured. To prove that i) $\iff$ iv) we refer to [3].

As previously mentioned, this result is no longer true when multivariate random variables are considered with $n \geq 3$ (see [7] and [8]).

3 Multivariate distorted risk measures

Distorted probabilities have been developed in the theory of risk to consider the hypothesis that the original probability is not adequate to describe the distribution (for example to protect us against some events), but another motivation can be finding the risk neutral density function to price derivatives in complete market setting. These probabilities generate new risk measures, called distorted risk measures (just to give some examples see [2], [13], [14]).

In this section we try to deepen our knowledge about distorted risk measures in the multidimensional case. Something about this topic is told in [9], but here there is not a representation through complete mathematical results.

To be clear we say that the definition of capacity still holds in the multivariate setting, but Choquet integral representation is not yet valid for random vectors, because we don’t know if it can be expressed as a Riemann integral in $\mathbb{R}^n$.

But we can define the distortion risk measure in the multivariate case as:

**Definition 3.1** Given a distortion $g$, that is a non decreasing function such that $g : [0, 1] \rightarrow [0, 1]$, with $g(0) = 0$ and $g(1) = 1$, a vector distorted risk measure is the functional:

$$R_g [X] = \int_0^{+\infty} \ldots \int_0^{+\infty} g(S^X(x)) \, dx_1 \ldots dx_n.$$  

We note that the function $g(S^X(x)) : \mathbb{R}_+^n \rightarrow [0, 1]$ is non increasing in each component.

**Proposition 3.1** The properties of the multivariate distorted risk measures are the following:

i) Monotonicity;

ii) Positive homogeneity of order $n$;

iii) Constancy;

iv) Translation invariance;
v) **Subadditivity with concave** $g$;

vi) **Superadditivity with convex** $g$;

vii) **Convexity with concave** $g$;

viii) **Non excessive loading**;

ix) **Expectation boundedness with concave** $g$.

**Proof:**

i) It follows from the relationship between multivariate stochastic orders: if $X \preceq_{SD} Y$, then $X \preceq_{LO} Y$ and $Y \preceq_{LO} X$, this implies that $F_X \leq F_Y$ and $S^X \geq S^Y$ and from the properties of distortion function $g$, follows that

$$R_g[Y] = \int_0^{+\infty} \ldots \int_0^{+\infty} g(S^Y(x)) \, dx_1 \ldots dx_n \geq \int_0^{+\infty} \ldots \int_0^{+\infty} g(S^X(x)) \, dx_1 \ldots dx_n = R_g[X].$$

ii) It is a consequence of the fact that $S^X(t) = S^X(t)$, so we have

$$R_g[cX] = \int_0^{+\infty} \ldots \int_0^{+\infty} g(S^X(t)) \, dt_n \ldots dt_1 = c^n \int_0^{+\infty} \ldots \int_0^{+\infty} g(S^X(u)) \, du_n \ldots du_1 = c^n R_g[X],$$

where $u = \frac{t}{c}$.

iii) $R_g[b] = \bar{b}$ with $b \in \mathbb{R}_+^n$, in fact $\int_0^{b_1} \ldots \int_0^{b_n} g(1) \, dt_2 \ldots dt_1 = b_n \ldots b_1 = \bar{b}$.

iv) We want to show that $R_g[X + a] = R_g[X] + \bar{a}$. We recall that $S^{X+a}(t) = S^X(t-a)$, and calling $t-a = u$ we have:

$$R_g[X + a] = \int_{-a_1}^{+\infty} \ldots \int_{-a_n}^{+\infty} g(S^X(u)) \, du_n \ldots du_1 = R_g[X] + \int_{-a_1}^{0} \ldots \int_{-a_n}^{0} 1 \, du_n \ldots du_1 = R_g[X] + \bar{a}.$$

v) To prove subadditivity we recall a definition of concavity: if $g$ is a concave function, then we have that $g(a + c) - g(a) \geq g(b + c) - g(b)$ with $a \leq b$ and $c \geq 0$. We apply this definition pointwise to $S^X \leq S^Y$ with $S^{X+Y} \geq 0$ to obtain $g(S^X + S^X + Y) - g(S^X) \geq g(S^{X+Y} + Y) - g(S^Y)$. We isolate $g(S^{X+Y})$ and we have $g(S^{X+Y}) \leq g(S^X) + g(S^Y)$, since $g(S^X + S^{X+Y}) \leq g(2S^X)$ from the subadditivity of the function $g$. The result follows integrating these components.

vi) It follows using a convexity requirement for $g$ and inverting the sign of inequalities.

vii) It is obvious from properties i), ii) and v).

viii) $R_g[X] \leq \sup_{\omega \in \Omega} \{ \{X_1(\omega)\}, \ldots, \{X_n(\omega)\} \}$, because if we suppose that $X$ is a vector of constant components, i.e $X = (c_1, \ldots, c_n)$ the distorted risk measure is $R[X] = c_1 \ldots c_n$ that is obviously smaller than $c_i^i$ where $c_i = \sup_{\omega \in \Omega} c_i, i = 1, \ldots, n$.

ix) $R_g[X] \geq E[X_1 \ldots X_n] = \int_{0}^{+\infty} \ldots \int_{0}^{+\infty} S^X(x) \, dx$ if $g$ is concave.
In the multivariate case doesn’t hold the equality $F_X = 1 - S_X$ and thus it is not in general true the relation $\int_{\mathbb{R}_+^n} g (S_X(x)) \, dx = \int_{\mathbb{R}_+^n} [1 - f (F_X(x))] \, dx$ with $f : [0, 1] \to [0, 1]$, increasing function.

Moreover doesn’t hold the duality relationship between the functions $f$ and $g$, thus is not in general true the equation $g(x) = 1 - f(1 - x)$. Our choice is to renounce to their duality to save the concept of distortion either of the survival function, or of the distribution function. Therefore we can observe the differences in the two different approaches.

**Definition 3.2** Given a distortion function $f : [0, 1] \to [0, 1]$, increasing and such that $f(0) = 0$ and $f(1) = 1$, a vector distorted risk measure is the functional:

$$R_f [X] = \int_{\mathbb{R}_+^n} [1 - f (F_X(x))] \, dx.$$ 

Now we have subadditivity with a convex function $f$ and this leads to the convexity of the measure $R_f$.

Remembering that a distortion is a univariate function even when we deal with random vectors and multivariate distributions we can also define vector Value at Risks and vector Conditional Value at Risks, using slight alterations of the usual distortions for VaR and CVaR respectively, and composing these with the multivariate tail distributions or the distribution functions.

**Definition 3.3** Let $X$ be a random vector that takes on values in $\mathbb{R}_+^n$ and $g$ be the distortion defined as:

$$g (S_X(x)) = \begin{cases} 
0 & 0 \leq S_{X_i}(x_i) \leq 1 - p_i \\
1 & 1 - p_i < S_{X_i}(x_i) \leq 1 
\end{cases}$$

*Vector Value at Risk* is the distorted measure

$$\text{VaR}[X; p] = \int_0^{+\infty} \ldots \int_0^{+\infty} g (S_X(x)) \, dx_1 \ldots dx_n,$$

expressed with the usual formula, using this distortion $g$ above defined.

If we want to give to this formulation a more explicit form we can consider the componentwise order for which $x > \text{VaR}[X; p]$ stands for $x_i > \text{VaR}[X_i; p]$ for all $i = 1, \ldots, n$ or more lightly $x_i > \text{VaR}_X$, and we can rewrite the distortion as:

$$g (S_X(x)) = \begin{cases} 
0 & x_i \geq \text{VaR}_{X_i} \\
1 & 0 \leq x_i < \text{VaR}_{X_i} 
\end{cases}$$

and we obtain

$$\text{VaR}[X; p] = \int_0^{\text{VaR}_X} \ldots \int_0^{\text{VaR}_X} 1 \, dx_1 \ldots dx_n = \text{VaR}_{X_1} \ldots \text{VaR}_{X_n}.$$ 

Obviously this result let us think that to consider a componentwise order is something like to consider an independency between the components of the random vector. Actually we are considering only the case in which the components are concordant.

In the same way we can define the vector Conditional Value at Risk:
\textbf{Definition 3.4} Let $X$ be a random vector with values in $\mathbb{R}^n_+$ and $g$ be the distortion function defined as:

$$g \left( S^X(x) \right) = \begin{cases} \frac{S^X(x)}{\prod_{i=1}^{n} (1-p_i)} & 0 \leq S^X_i(x_i) \leq 1 - p_i \\ 1 - p_i < S^X_i(x_i) \leq 1 \end{cases}$$

Vector CVaR is the distorted measure

$$CVaR[X;p] = \int_0^{\infty} \ldots \int_0^{\infty} g \left( S^X(x) \right) \, dx_1 \ldots dx_n,$$

expressed using the distortion $g$ above defined.

A more tractable form is given by:

$$g \left( S^X(x) \right) = \begin{cases} \frac{S^X(x)}{\prod_{i=1}^{n} (1-p_i)} & x_i \geq VaR_{X_i} \\ 1 - p_i < x_i < VaR_{X_i} \end{cases}$$

we arrive to this formula:

$$CVaR[X;p] = \int_0^{VaR_{X_1}} \ldots \int_0^{VaR_{X_n}} 1 \, dx_1 \ldots dx_n + \int_{VaR_{X_1}}^{\infty} \ldots \int_{VaR_{X_n}}^{\infty} \frac{S^X(x)}{\prod_{i=1}^{n} (1-p_i)} \, dx_1 \ldots dx_n = VaR[X;p] + \int_{VaR_{X_1}}^{\infty} \ldots \int_{VaR_{X_n}}^{\infty} \frac{S^X(x)}{\prod_{i=1}^{n} (1-p_i)} \, dx_1 \ldots dx_n.$$

The second part of the formula is not easy to render explicitly if we don’t introduce an independence hypothesis.

If we follow Definition 3.2 instead of 3.1 we can introduce a different formulation for CVaR, very useful in proving a nice result proposed later on.

The increasing convex function $f$ used in the definition of CVaR is the following:

$$f \left( F^X(x) \right) = \begin{cases} \frac{F^X(x) - 1 + \sum_{i=1}^{n} (1-p_i)}{\prod_{i=1}^{n} (1-p_i)} & 0 \leq F^X_i(x_i) < p_i \\ F^X_i(x_i) & F^X_i(x_i) \geq p_i \end{cases}$$

\textbf{Definition 3.5} Let $X$ be a random vector that takes on values in $\mathbb{R}^n_+$ and $f$ be an increasing function $f : [0,1] \to [0,1]$, such that $f(0) = 0$ and $f(1) = 1$ and defined as above. The Conditional Value at Risk distorted by such a $f$ is the following:

$$CVaR[X;p] = \int_0^{VaR_{X_1}} \ldots \int_0^{VaR_{X_n}} 1 \, dx_1 \ldots dx_n + \int_{VaR_{X_1}}^{\infty} \ldots \int_{VaR_{X_n}}^{\infty} \frac{F^X(x)-1+\sum_{i=1}^{n} (1-p_i)}{\prod_{i=1}^{n} (1-p_i)} \, dx_1 \ldots dx_n.$$

We recall here that if $0 \leq x_i < VaR_{X_i}$, or $F^X_i < p_i$, then $F^X < \min_i \{p_i\}$, while if $x_i \geq VaR_{X_i}$, or $F^X_i \geq p_i$, then $F^X \geq \max_i \{\sum_{i=1}^{n} p_i - (n-1), 0\}$. Therefore we have that if $F^X \leq \min_i \{p_i\}$ also $F^X \leq 1 - \prod_{i=1}^{n} (1-p_i)$. This let us consider also the bounds for the joint distribution, not only for the marginals. Finally we can present an interesting result about the representation of subadditive distorted risk measures through convex combinations of Conditional Value at Risks.
Theorem 3 Let \( X \in \mathcal{X} \). Consider a subadditive multivariate distortion in the form \( R_f [X] = \int_{\mathbb{R}^n_+} [1 - f (F^X (x))] \, dx \). Then there exists a probability measure \( \mu \) on \([0, 1]\) such that:

\[
R_f [X] = \int_0^1 CVaR [X; p] \, d\mu (p).
\]

Proof: A multivariate distorted measure of this kind \( R_f [X] = \int_{\mathbb{R}^n_+} [1 - f (F^X (x))] \, dx \) is subadditive if \( f \) is a convex, increasing function such that: \( f : [0, 1] \rightarrow [0, 1] \) with \( f(0) = 0 \) and \( f(1) = 1 \).

Let \( p = 1 - \prod_{i=1}^{n} (1 - p_i) \), then a probability measure \( \mu (p) \) exists such that this function \( f \) can be represented as: \( f (u) = \int_0^1 \frac{(u-p)}{(1-p)} \, d\mu (p) \) with \( p \in [0, 1] \).

Then we can write \( \forall X \in \mathcal{X} \) (through Fubini theorem):

\[
R_f [X] = \int_{\mathbb{R}^n_+} [1 - f (F^X (x))] \, dx = \int_{\mathbb{R}^n_+} \left[ 1 - \int_0^{1 \cdots 1} \frac{(F^X (x) - 1 + \prod_{i=1}^{n} (1 - p_i))_{+}}{\prod_{i=1}^{n} (1 - p_i)} \right] \, d\mu (p) \, dx = \\
\int_{\mathbb{R}^n_+} \left[ 1 - \int_0^1 \frac{(F^X (x) - 1 + \prod_{i=1}^{n} (1 - p_i))_{+}}{\prod_{i=1}^{n} (1 - p_i)} \right] \, d\mu (p) = \\
\int_0^1 d\mu (p) \int_{\mathbb{R}^n_+} \left[ 1 - \frac{(F^X (x) - 1 + \prod_{i=1}^{n} (1 - p_i))_{+}}{\prod_{i=1}^{n} (1 - p_i)} \right] \, dx = \int_0^1 CVaR [X; p] \, d\mu (p).
\]

We easily understand how all Decision Makers who want to investigate the riskiness of their portfolio through multivariate distorted risk measures, and thus through enveloping the probability given by the market, are, in fact, evaluating all the possible CVaRs of the portfolio as long as \( p \)-levels vary. This exactly reflects the same considerations that could be done in the scalar case for a single investment, even if we cannot say that all the possible multivariate distorted risk measures can be interpreted as a Choquet integral.

3.1 Bivariate case

Since not every result about stochastic dominance works in multivariate setting, we restrict our attention to the bivariate one. Anyway, it is interesting because it takes into consideration the riskiness not only of the marginal distributions, but also of the joint distribution, tracing out a course of action to multivariate generalizations. It is worth noting that this procedure has something to do with concordance measures (or measures of dependence) that we will describe later on.

We start from some observations about VaR and CVaR formulated through distorted probabilities. Letting \( X \) be a random vector with values in \( \mathbb{R}^2_+ \), we have:

\[
VaR [X; p] = VaR_{X_1} VaR_{X_2}
\]

and

\[
CVaR [X; p] = VaR_{X_1} VaR_{X_2} + \int_{VaR_{X_2}}^{+\infty} \int_{VaR_{X_1}}^{+\infty} \frac{S^X (x_1, x_2)}{(1 - p_1) (1 - p_2)} \, dx_1 dx_2.
\]

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Under independence hypothesis we know that $S^X(x) = S^{X_1}(x_1) S^{X_2}(x_2)$ and we then rewrite CVaR in the following way:

$$CVaR \{X; \alpha\} = \text{VaR}_{X_1} \text{VaR}_{X_2} + \frac{1}{(1-p_1)(1-p_2)} \int_{\text{VaR}_{X_2}}^{+\infty} \int_{\text{VaR}_{X_1}}^{+\infty} S^{X_1}(x_1) S^{X_2}(x_2) \, dx_1 dx_2.$$ 

Let us now to concentrate on $\int_{\text{VaR}_{X_2}}^{+\infty} \int_{\text{VaR}_{X_1}}^{+\infty} S^{X_1}(x_1) S^{X_2}(x_2) \, dx_1 dx_2$. We have

$$\int_{\text{VaR}_{X_2}}^{+\infty} \int_{\text{VaR}_{X_1}}^{+\infty} S^{X_1}(x_1) S^{X_2}(x_2) \, dx_1 dx_2 = \int_{\text{VaR}_{X_2}}^{+\infty} S^{X_2}(x_2) \, dx_2 \int_{\text{VaR}_{X_1}}^{+\infty} S^{X_1}(x_1) \, dx_1 = \int_{\text{VaR}_{X_1}}^{+\infty} S^{X_1}(x_1) \left[ - (1 - p_1) \text{VaR}_{X_1} - \int_{\text{VaR}_{X_1}}^{+\infty} x_1 dS^{X_1}(x_1) \right] \, dx_2;$$

we have integrated by part the first integral, and repeating the procedure for the second integral we obtain

$$(1 - p_1)(1 - p_2) \text{VaR}_{X_1} \text{VaR}_{X_2} + (1 - p_1) \text{VaR}_{X_1} \int_{\text{VaR}_{X_2}}^{+\infty} x_2 dS^{X_2}(x_2) + (1 - p_2) \text{VaR}_{X_2} \int_{\text{VaR}_{X_1}}^{+\infty} x_1 dS^{X_1}(x_1) \int_{\text{VaR}_{X_2}}^{+\infty} x_2 dS^{X_2}(x_2),$$

that leads, together with the first part, to:

$$CVaR \{X; \alpha\} = 2\text{VaR}_{X_1} \text{VaR}_{X_2} - \text{VaR}_{X_1} E[X_2 | X_2 > \text{VaR}_{X_2}] - \text{VaR}_{X_2} E[X_1 | X_1 > \text{VaR}_{X_1}] + E[X_2 | X_2 > \text{VaR}_{X_2}] E[X_1 | X_1 > \text{VaR}_{X_1}].$$

This result holds only if the vector has independent components, but try to remember it, because it will be interesting recall it soon, dealing with a new multivariate measure.

### 4 Measures of concordance

In the previous section we have introduced the well-known stochastic ordering of concordance where concordance between two random variables arises if large values of one variable tend to occur with large values of the other and small values occur with small values of the other. So concordance considers nonlinear associations between random variables that correlation might miss. Now, we want to consider the main characteristics a measure of concordance should have, as well as to the definition of some classes of measures of integral form. We restrict our attention to the bivariate case.

In 1984 Scarsini ([10]) defined a set of axioms that a bivariate dependence ordering of distributions should have in order that higher in the ordering means more positive concordance. By a measure of concordance we mean a function that attaches to every continuous bivariate random vector a real number $\alpha(X_1, X_2)$ satisfying the following properties:

1. $-1 \leq \alpha(X_1, X_2) \leq 1$;
2. $\alpha(X_1, X_1) = 1$;
3. $\alpha(X_1, -X_1) = -1$;
4. $\alpha(-X_1, X_2) = \alpha(X_1, -X_2) = -\alpha(X_1, X_2)$;
5. $\alpha(X_1, X_2) = \alpha(X_2, X_1)$;
6. if $X_1$ and $X_2$ are independent, then $\alpha(X_1, X_2) = 0$;

7. if $(X_1, X_2) \preceq_C (Y_1, Y_2)$ then $\alpha(X_1, X_2) \leq \alpha(Y_1, Y_2)$

8. if $\{X\}_n$ is a sequence of bivariate random vectors converging in distribution to $X$, then $\lim_{n \to \infty} \alpha(X_n) = \alpha(X)$.

Now we consider the dihedral group $D_4$ of the symmetries on the square $[0,1]^2$. We have $D_4 = \{e, r, r^2, r^3, h, hr, hr^2, hr^3\}$ where $e$ is the identity, $h$ is the reflection about $x = \frac{1}{2}$, and $r$ is a $90^\circ$ counterclockwise rotation.

A measure $\mu$ on $[0,1]^2$ is said to be $D_4$-invariant if its value for any Borel set $A$ of $[0,1]^2$ is invariant with respect to the symmetries of the unit square that is $\mu(A) = \mu(d(A))$.

**Proposition 4.1** If $\mu$ is a bounded $D_4$-invariant measure on $[0,1]^2$, there exist $\alpha, \beta \in \mathbb{R}$ such that the function defined by

$$
\rho((X_1, X_2)) = \alpha \int_{[0,1]^2} F^{X_1}(x_1, x_2) d\mu(F^{X_1}(x_1), F^{X_2}(x_2)) - \beta
$$

is a concordance measure.

**Proof:** A measures of concordance associates to a continuous bivariate random vector depends only on the copula associate to the vector since a measure of concordance is invariant under invariant increasing transformation of the random variables. So the result follows from Theorem 3.1 of ([4])

\[\Box\]

5. A vector-valued measure

We know that

$$
E \left[(X - VaR_X)_+\right] = \int_{VaR_X}^{+\infty} (x - VaR_X) dF^X(x) = \int_{VaR_X}^{+\infty} S^X(x) dx
$$

is also called stop-loss premium with retention $VaR_X$. In Definition 2.6 we have introduced the concept of product stop-loss transform for random vectors, we use this approach to give a definition for a new measure that we call Product Stop-loss Premium.

**Definition 5.1** Consider a non-negative bivariate random vector $X$ and calculate Value at Risk of its single components. Product Stop-loss Premium is defined as follows:

$$
PSP[X; p] = E \left[ (X_1 - VaR_{X_1})_+ (X_2 - VaR_{X_2})_+ \right].
$$

Of course this definition could be extended also in general case, writing:

$$
PSP[X; p] = E \left[ (X_1 - VaR_{X_1})_+ \cdots (X_n - VaR_{X_n})_+ \right],
$$
but some properties will be different, because not everything stated for the bivariate case works in the multivariate one.

This measure could also represent a kind of Tail Conditional Covariance for the multidimensional case. Our aim is to give a multivariate measure that can detect the joint tail risk of the distribution. In doing this we also have a representation of the marginal risks and thus the result is a measure that describes the joint and marginal risk in a simple and intuitive manner.

We examine in particular the case $X_1 > VaR_{X_1}$ and $X_2 > VaR_{X_2}$ simultaneously, since large and small values will tend to be more often associated under the distribution which dominates the other one.

Random variables are concordant if they tend to be all large together or small together and in this case we have a measure with non trivial values when the variables exceed given thresholds together and are not constant, otherwise we have $PSP[X; p] = 0$.

It is clear that concordance affects this measure, and in general we know that concordance behaviour influences risk management of large portfolios of insurance contracts or financial assets. In these portfolios the main risk is the occurrence of many joint default events or simultaneous downside evolutions of prices.

The approach is closely related to dependence measures and then to copula. We obtain in few passages:

$$PSP[X; p] = E[(X_1 - VaR_{X_1})_+ (X_2 - VaR_{X_2})_+ ] = \int_{VaR_{X_1}}^{\infty} \int_{VaR_{X_2}}^{\infty} S^X(x) dx_1 dx_2 - VaR_{X_1} E[X_1 | X_1 > VaR_{X_1}] - VaR_{X_1} E[X_2 | X_2 > VaR_{X_2}] + VaR_{X_1} VaR_{X_2}.$$  

PSP for multivariate distributions is interpreted as a measure that can keep the dependence structure of the components of the random vector considered, when specified thresholds are exceeded by each component with probability $p_i$; but indeed it is also a measure that can evaluate the joint as well as the marginal risk.

Moreover with distorted risk measures in the bivariate independent case and considering the case $X_1 > VaR_{X_1}$ and $X_2 > VaR_{X_2}$, we obtain this risk measure:

$$CVaR[X; p] = E[X_1 | X_1 > VaR_{X_1}] E[X_2 | X_2 > VaR_{X_2}] - VaR_{X_1} E[X_2 | X_2 > VaR_{X_2}] - VaR_{X_1} E[X_1 | X_1 > VaR_{X_1}] + VaR_{X_1} VaR_{X_2},$$

while with our PSP under the same restrictions we have:

$$PSP[X; p] = \int_{VaR_{X_1}}^{\infty} \int_{VaR_{X_2}}^{\infty} S^{X_1}(x_1)S^{X_2}(x_2) dx_1 dx_2 - VaR_{X_2} E[X_1 | X_1 > VaR_{X_1}] - VaR_{X_1} E[X_2 | X_2 > VaR_{X_2}] + VaR_{X_1} VaR_{X_2} = E[X_1 | X_1 > VaR_{X_1}] E[X_2 | X_2 > VaR_{X_2}] - VaR_{X_1} E[X_2 | X_2 > VaR_{X_2}] - VaR_{X_1} E[X_2 | X_2 > VaR_{X_2}] + VaR_{X_1} VaR_{X_2}.$$

We can conclude that these risk measures are the same for bivariate vectors with independent components, conditioning on the case $X_1 > VaR_{X_1}$ and $X_2 > VaR_{X_2}$. 

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Stochastic orderings are binary relations defined on classes of probability distributions. They aim to mathematically translate intuitive ideas like “being larger” or “being more variable” for random quantities. They thus extend the classical mean-variance approach to compare riskiness. We propose here a way to compare dependence introducing a stochastic order based on our PSP measure (Definition 5.1), and stressing that higher dependence leads to higher riskiness.

**Proposition 5.1** If \( X, Y \in \mathcal{R}_2 \), then holds:

\[
X \preceq_{SM} Y \iff \text{PSP}[X; p] \leq \text{PSP}[Y; p] \quad \forall p.
\]

**Proof:** If \( X \preceq_{SM} Y \) then \( E[f(X)] \leq E[f(Y)] \) for every supermodular function \( f \), therefore also for the specific supermodular function that defines our PSP and then follows \( \text{PSP}[X; p] \leq \text{PSP}[Y; p] \). Conversely if \( \text{PSP}[X; p] \leq \text{PSP}[Y; p] \) and \( X, Y \in \mathcal{R}_2 \), we have

\[
\int_{VaR_X_2}^{+\infty} \int_{VaR_X_1}^{+\infty} S^X(t) \, dt \leq \int_{VaR_Y_2}^{+\infty} \int_{VaR_Y_1}^{+\infty} S^Y(t) \, dt
\]

with \( VaR_X_2 = VaR_Y_2 \) and \( VaR_X_1 = VaR_Y_1 \). It follows that \( S^X(t) \leq S^Y(t) \) that leads to \( X \preceq_{C} Y \). From Theorem 1 follows \( X \preceq_{SM} Y \). \( \square \)

Obviously PSP is also consistent with the concordance order.

Risk measures that are subadditive for all possible dependence structures of the vectors do not reflect the dependence between \((X_1 - \alpha_1)_+\) and \((X_2 - \alpha_2)_+\).

We can note that our PSP is not always subadditive, in fact if we take the non-negative vectors \( X, Y \in \mathcal{X} \) we should have:

\[
E \left[ (X_1 + Y_1 - VaR_X_1 - VaR_Y_1)_+ (X_2 + Y_2 - VaR_X_2 - VaR_Y_2)_+ \right] \leq E \left[ (X_1 - VaR_X_1)_+ (X_2 - VaR_X_2)_+ \right] + E \left[ (Y_1 - VaR_Y_1)_+ (Y_2 - VaR_Y_2)_+ \right].
\]

After verifying all the possible combinations among scenarios \( X_1 > VaR_X_1, X_1 < VaR_X_1, X_2 > VaR_X_2, X_2 < VaR_X_2, Y_1 < VaR_Y_1, Y_2 > VaR_Y_2, Y_2 < VaR_Y_2 \), we can conclude that

- the components of one vector are discordant while that one of the other are concordant and it happens \( X_i + Y_i > VaR_X_i + VaR_Y_i \) and \( X_j + Y_j > VaR_X_j + VaR_Y_j \)
- the components of both vectors are concordant but \( X_i > (>) VaR_X_i \) and \( Y_i < (>) VaR_Y_i \) \( \forall i \) and also happens \( X_i + Y_i > VaR_X_i + VaR_Y_i \) \( \forall i \)
- \( X_i > VaR_X_i \) and \( Y_i > VaR_Y_i \) \( \forall i \) simultaneously.
References


