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Martina Nardon and Paolo Pianca

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Simulation techniques for generalized Gaussian densities

Martina Nardon <mnardon@unive.it>

Paolo Pianca <pianca@unive.it>

Department of Applied Mathematics
University Ca' Foscari Venice

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Abstract. This contribution deals with Monte Carlo simulation of generalized Gaussian random variables. Such a parametric family of distributions has been proposed in many applications in science to describe physical phenomena and in engineering, and it seems also useful in modeling economic and financial data. For values of the shape parameter α within a certain range, the distribution presents heavy tails. In particular, the cases $\alpha = 1/3$ and $\alpha = 1/2$ are considered. For such values of the shape parameter, different simulation methods are assessed.

Keywords: Generalized Gaussian density, heavy tails, transformations of random variables, Monte Carlo simulation, Lambert W function.

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Correspondence to:

Martina Nardon	Department of Applied Mathematics
	University Ca' Foscari Venice
	Dorsoduro 3825/E
	30123 Venice, Italy
Phone:	[++39] 041 2346924
Fax:	[++39] 041 5221756
E-mail:	mnardon@unive.it

1 Introduction

The parametric family of generalized Gaussian distributions has been used in science to model successfully physical phenomena and in engineering as, for example, in the area of signal processing and in audio and video encoding (see, e.g., Miller and Thomas, 1972, and Kokkinakis and Nandi, 2005).

Some special important classes of random variables belong to this family such as, amongst others, the Gaussian distribution and the Laplace distribution. Moreover, for values of the shape parameter within a certain range, which is of interest in many practical applications, the distribution presents heavy tails.

The generalized Gaussian distribution seems also useful in finance: there is evidence of heavy tails in financial high-frequency data (see Müller *et al.*, 1998). For instance, the process of financial asset returns can be modeled by a generalized Gaussian noise.

In most practical situations accurate and fast simulation of a stochastic process of interest can play an important role. In this contribution, we analyze some techniques for generating deviates from a generalized Gaussian distribution.

In particular, we address the cases $\alpha = 1/3$ and $\alpha = 1/2$. For such values of the shape parameter, different simulation methods are assessed. In the special case $\alpha = 1/2$, we compare the efficiency of four different simulation techniques. Numerical results highlight that the technique based on the inverse cumulative distribution function written in terms of the Lambert W function is the most efficient.

The Lambert W function, also known as the *Omega* function, is a multivalued complex function defined as the inverse of the function $f(w) = w\,e^w$. It has many applications in pure and applied mathematics (see Corless *et al.*, 1996). We briefly review the main properties of this special function.

An outline of the paper is the following. Section 2 presents some properties of the generalized Gaussian density. In sections 3 and 4 simulation techniques for a generalized Gaussian random variables are analyzed. In particular, the choices of $\alpha = 1/3$ and $\alpha = 1/2$ are considered. Section 5 concludes. Appendix A reports the proof of a result on transformations of random variables. Appendix B concerns the simulation of a gamma random variable.

2 The generalized Gaussian density

The probability density function (pdf) of a generalized Gaussian random variable X, with mean μ and variance σ^2, is defined as

$$f_X(x;\,\mu,\sigma,\alpha) = \frac{\alpha}{2}\frac{A(\alpha,\sigma)}{\Gamma(1/\alpha)}\,\exp\left\{-\left(A(\alpha,\sigma)\,|\,x-\mu\,|\,\right)^{\alpha}\right\} \qquad x \in \mathbb{R}\,, \tag{1}$$

where

$$A(\alpha,\sigma) = \frac{1}{\sigma}\left[\frac{\Gamma(3/\alpha)}{\Gamma(1/\alpha)}\right]^{1/2} \tag{2}$$

and

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \qquad z > 0 \tag{3}$$

is the complete gamma function.

The generalized Gaussian distribution (GGD) is symmetric with respect to μ. $A(\alpha, \sigma)$ is a scaling factor which defines the dispersion of the distribution, hence it is a generalized measure of the variance. $\alpha > 0$ is the shape parameter which describes the exponential rate of decay: heavier tails correspond to smaller values of α.

The generalized Gaussian family includes a variety of random variables. Some well known classes of distributions are generated by a parametrization of the exponential decay of the GGD. When $\alpha = 1$, the GGD corresponds to a Laplacian, or double exponential, distribution. For $\alpha = 2$ one has a Gaussian distribution. When $\alpha \to +\infty$ the GGD converges to a uniform distribution in $(\mu - \sqrt{3}\sigma,\ \mu + \sqrt{3}\sigma)$, while when $\alpha \to 0^+$ we have an impulse probability function at $x = \mu$.

All the odd central moments of distribution (1) are zero, $\mathbb{E}(X - \mu)^r = 0$ $(r = 1, 3, 5, \ldots)$, and the even central moments are

$$\mathbb{E}(X - \mu)^r = \left[\frac{\sigma^2 \Gamma(1/\alpha)}{\Gamma(3/\alpha)}\right]^{r/2} \frac{\Gamma((r+1)/\alpha)}{\Gamma(1/\alpha)} \qquad r = 2, 4, 6, \ldots \tag{4}$$

With a straightforward standardization and some reductions from (1), we obtain a GG random variable with zero-mean and unit-variance having the following density

$$f_X(x;\, \alpha) = \frac{\alpha}{2} \frac{A(\alpha)}{\Gamma(1/\alpha)} \exp\left\{-\left(A(\alpha)\,|\,x\,|\right)^{\alpha}\right\}, \tag{5}$$

where $A(\alpha) = A(\alpha, 1)$.

In the following, we confine our attention to generalized Gaussian random variables with density (5). For $0 < \alpha < 2$ the density (5) is suitable for modeling many physical and financial processes with heavy tails. Moreover, it is worth noting that, all the moments are finite (this is not the case for other heavy-tailed densities, like e.g. stable densities).

The kurtosis of distribution (5) is

$$\mathcal{K}(\alpha) = \frac{\Gamma(1/\alpha)\,\Gamma(5/\alpha)}{\left[\Gamma(3/\alpha)\right]^2}. \tag{6}$$

$\mathcal{K}(\alpha)$ decreases with α; moreover the following results hold:

$$\lim_{\alpha \to 0^+} \mathcal{K}(\alpha) = +\infty \qquad \lim_{\alpha \to +\infty} \mathcal{K}(\alpha) = 1.8 \tag{7}$$

(see Domínguez-Molina and González-Farías, 2002). Figure 1 shows the generalized Gaussian densities for different values of the parameter α, with zero mean and unit variance. Figure 2 shows the kurtosis behavior for $\alpha \in [0,\ 2]$ and $\alpha \in [0,\ 20]$.

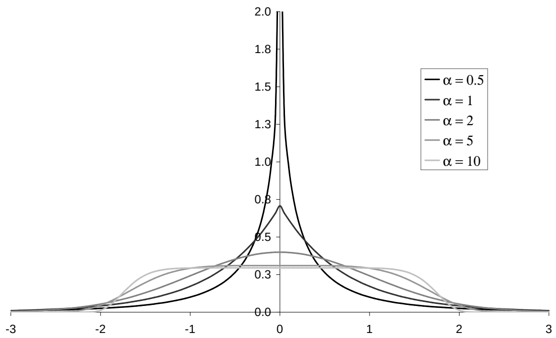
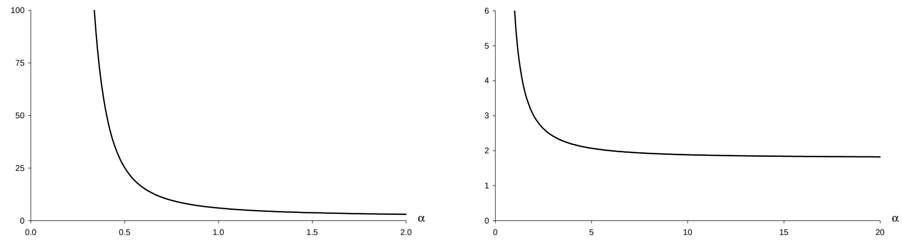


Figure 1: Generalized Gaussian densities for different values of the parameter α, with $\mu = 0$ and $\sigma = 1$.



(a) (b)

Figure 2: Kurtosis of unit variance GGD as α varies, with $\alpha \in [0,\, 2]$ in case (a), and $\alpha \in [0,\, 20]$ in case (b).

3 Simulating the generalized Gaussian distribution

Let $F_X(\cdot)$ be the cumulative distribution function (cdf) of a random variable X and $F_X^{-1}(\cdot)$ be its inverse. It is well known that if $F_X^{-1}(\cdot)$ can be directly evaluated, a large number of realizations of X can be obtained as $x_i = F_X^{-1}(u_i)$, where u_i $(i = 1, 2, \ldots, n)$ are random numbers uniform over $(0, 1)$. If F_X^{-1} has a closed form expression such a method can be applied efficiently; unfortunately, this is not always the case. Nevertheless, if $F_X(\cdot)$ can be evaluated, it can still be possible to simulate the random variable X by numerically inverting its cumulative distribution function. When $F_X(\cdot)$ has no closed form expression, numerical integration or other approximation methods are required, at the expense of an increasing computational amount. An alternative simulation method is based on techniques of transformation of a random variable X for which a random number generator is available. A specific technique will be described below for the GGD case.

Let X be a generalized Gaussian random variable with cumulative distribution function

$$F_X(x) = \int_{-\infty}^{x} \frac{\alpha}{2} \frac{A(\alpha)}{\Gamma(1/\alpha)} \exp\left\{- \left(A(\alpha)\,|\,t\,|\,\right)^{\alpha}\right\} dt\,, \tag{8}$$

where $A(\alpha)$ has been defined above. Such a function can be written in closed form only in a very few special cases.

In order to generate values from a generalized Gaussian distribution with parameter α, the following three-step procedure can be used:

i) simulate a gamma random variable $Z \sim Gamma(a, b)$, with parameters $a = \alpha^{-1}$ and $b = (A(\alpha))^{\alpha}$;

ii) first apply the transformation $Y = Z^{1/\alpha}$;

iii) finally, apply a transformation of the form

$$Y = |\,X\,|\,. \tag{9}$$

Relationship (9) has two roots. The problem is how to determine the probability for choosing one of such roots. It can be shown[1] that, as a consequence of the symmetry of the GG distribution, one takes the roots with equal probability. For each random observation y, a root is chosen ($x = -y$ or $x = y$). To this end, an auxiliary Bernoulli trial with probability $p = 1/2$ can be performed[2].

We first encountered the problem of generating random variates from the gamma distribution[3]. The process relies on the assumption that if Z is a $Gamma(a, b)$ distributed random variable (with a and b as defined above), then by letting $Y = Z^{1/\alpha}$ and considering the transformation (9), as a result X has a GGD distribution with parameter α.

In testing the above procedure, a number of large samples were generated, with various choices of the parameter α. The Kolmogorov-Smirnov test was applied to each simulation experiment and yielded no evidence to indicate that the simulated observations were not generated from the GGD distribution.

3.1 Simulating a generalized Gaussian random variable with $\alpha = 1/3$

As a special case, we studied the generalized Gaussian density with $\alpha = 1/3$, while the case $\alpha = 1/2$ is analyzed in the following section.

When $\alpha = 1/3$, the density (5) becomes

$$f_X(x) = 2\sqrt{35}\,\exp\left\{-\left(24\sqrt{35}\,|\,x\,|\right)^{1/3}\right\} \tag{10}$$

[1] A detailed proof, based on a result of Michael *et al.* (1976), is given in the appendix.

[2] In practice, one does not need generating such an auxiliary variable. An instance of the antithetic variates technique can be applied by taking both roots, $x = -y$ and $x = y$, as realizations of a GGD.

[3] In the numerical experiments we used the routine DRNGAM of the IMSL library for the generation of deviates from a distribution $Gamma(a, 1)$. All the routines are coded in Fortran.

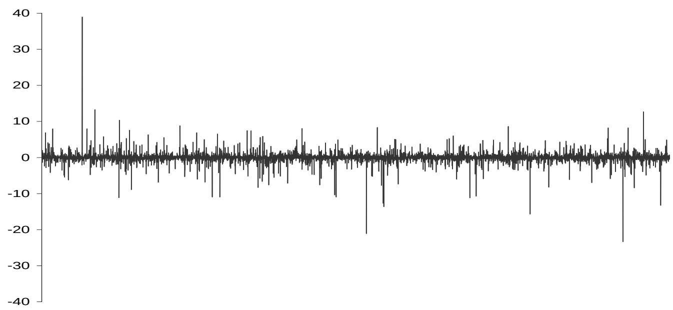


Figure 3: Instances of GGD with parameter $\alpha = 1/3$, $\mu = 0$, and $\sigma = 1$ ($N = 10\,000$).

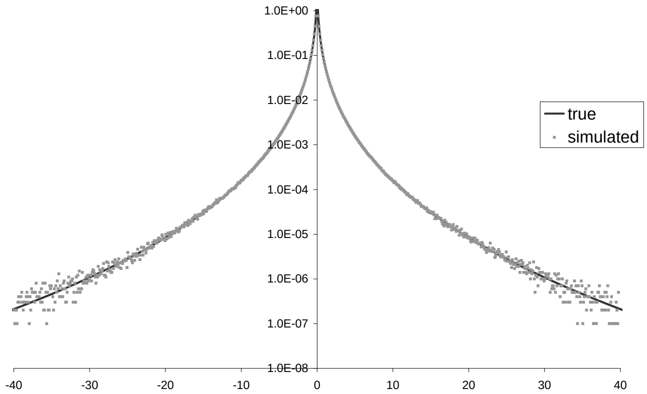


Figure 4: Estimated and theoretical probability density functions of a generalized Gaussian random variable with $\alpha = 1/3$ and kurtosis $\mathcal{K}(1/3) = 107.25$.

and the cumulative distribution function has the closed form

$$F_X(x) = \begin{cases} \frac{1}{2}\exp\left\{-\left(24\sqrt{35}\,(-x)\right)^{1/3}\right\} \\ \qquad \cdot\left(1 + \left(24\sqrt{35}\,(-x)\right)^{1/3} + \frac{1}{2}\left(24\sqrt{35}\,(-x)\right)^{2/3}\right) & x \leq 0 \\ 1 - \frac{1}{2}\exp\left\{-\left(24\sqrt{35}\,x\right)^{1/3}\right\} \\ \qquad \cdot\left(1 + \left(24\sqrt{35}\,x\right)^{1/3} + \frac{1}{2}\left(24\sqrt{35}\,x\right)^{2/3}\right) & x > 0\,. \end{cases} \tag{11}$$

3.2 Numerical results ($\alpha = 1/3$)

Figure 3 shows realizations of generalized Gaussian random variable with $\alpha = 1/3$, generated with the algorithm described above. We have performed an estimation of the probability

density function (10) based on 10^8 simulated values, gathered into intervals of width $\Delta x = 0.05$. Figure 4 shows the simulated probabilities and the theoretical density (10). As can be observed, the fit is fairly accurate.

We also compared the simulation procedure based on the gamma transformation with the method based on the inverse of the cdf (11)[4]. Numerical results show that the fit of the theoretical density is still good, but the method is computationally inefficient with respect to the simulation procedure based on the gamma transformation, being about ten times slower.

4 Simulating a generalized Gaussian random variable with $\alpha = 1/2$

In this section, the special case $\alpha = 1/2$ is considered. We compare the efficiency of four different simulation techniques. Numerical results highlight that the technique based on the inverse cumulative distribution function written in terms of the Lambert W function is the most efficient.

The Lambert W function is a multivalued complex function (see Corless *et al.*, 1996). We briefly review the main properties of such a special function in subsection 4.2.

4.1 The generalized Gaussian density with $\alpha = 1/2$

With regard to density (5), if we address the case $\alpha = 1/2$ we obtain the GG density

$$f_X(x) = \frac{\sqrt{30}}{2} \exp\left\{-2\sqrt{30}\,|\,x\,|^{1/2}\right\} \tag{12}$$

and the cumulative distribution function

$$F_X(x) = \begin{cases} \dfrac{1}{2}\exp\left\{-2\sqrt{30}\,(-x)^{1/2}\right\}\left(1 + 2\sqrt{30}\,(-x)^{1/2}\right) & x \leq 0 \\ \\ 1 - \exp\left\{-2\sqrt{30}\,x^{1/2}\right\}\dfrac{1}{2}\left(1 + 2\sqrt{30}\,x^{1/2}\right) & x > 0\,. \end{cases} \tag{13}$$

In order to generate variates from a GG distribution, we can apply the method based on the transformation described in section 3. As an alternative, realizations of a random variable with cdf (13) can be obtained as $x = F_X^{-1}(u)$, where u are random numbers from a uniform distribution on $(0, 1)$. As in the case $\alpha = 1/3$, the function $F_X(\cdot)$ can be inverted using numerical techniques.

The inverse function of the cdf (13) can also be expressed in terms of the so called *Lambert W function*. Based on such a result, a very fast and accurate simulation procedure can be defined. The following subsection is devoted to the definition of the Lambert W function and the description of its main properties.

[4] One should be careful when inverting $F_X(x)$. In particular, note that the success of the numerical algorithm depends on the initializing procedure.

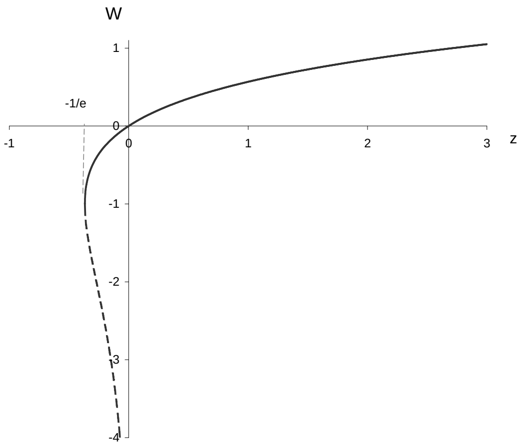


Figure 5: The two real branches of the Lambert W function: the principal branch $W_0(z)$ (solid line) and the branch $W_{-1}(z)$ (dashed line), $z \in [-1/e, +\infty)$.

4.2 The Lambert W function

The Lambert W function, specified implicitly as the root of the equation

$$W(z)e^{W(z)} = z\,, \tag{14}$$

is a multivalued function defined in general for z complex and assuming values $W(z)$ complex. If z is real and $z < -1/e$, then $W(z)$ is multivalued complex. If $z \in \mathbb{R}$ and $-1/e \leq z < 0$, there are two possible real values of $W(z)$: the branch satisfying $W(z) \geq -1$ is usually denoted by $W_0(z)$ and called the *principal branch* of the W function, and the other branch satisfying $W(z) \leq -1$ is denoted by $W_{-1}(z)$. If $z \in \mathbb{R}$ and $z \geq 0$, there is a single value for $W(z)$ which also belongs to the principle branch $W_0(z)$. The choice of solution branch usually depends on physical arguments or boundary conditions. The paper of Corless *et al.* (1996) reports the main properties and applications of such a function.

Figure 5 shows both the principal branch W_0 (solid line) and the branch denoted by W_{-1} (dashed line) of the Lambert W function.

A high-precision evaluation of the Lambert W function is available in *Maple* and *Mathematica* softwares. In particular, *Maple* computes the real values of W using the third-order Halley's method (see Alefeld, 1981),

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j) - \dfrac{1}{2} f''(x_j) \dfrac{f(x_j)}{f'(x_j)}} \qquad j \geq 0\,, \tag{15}$$

which in our case ($we^w = z$) gives rise to the following recursion

$$w_{j+1} = w_j - \frac{w_j\, e^{w_j} - z}{(w_j + 1)\, e^{w_j} - \dfrac{(w_j + 2)(w_j\, e^{w_j} - z)}{2w_j + 2}} \qquad j \geq 0\,. \tag{16}$$

Analytical approximations of the W function are also available and can be used as an initial guess in the iterative scheme (16) (see Corless *et al.*, 1996, and Chapeau-Blondeau and Monir, 2002).

In the numerical experiments carried out, we applied Halley's method and as initial guess we adopted

$$w_0 = \begin{cases} -1 + \rho - \frac{1}{3}\rho^2 + \frac{11}{72}\rho^3 - \frac{43}{540}\rho^4 + \frac{769}{17\,280}\rho^5 - \frac{221}{8\,505}\rho^6 & -\frac{1}{e} \le z < -0.333 \\ \frac{-8.096 + 391.0025z - 47.4252z^2 - 4877.633z^3 - 5532.776z^4}{1 - 82.9423z + 433.8688z^2 + 1515.306z^3} & -0.333 \le z \le -0.033 \\ L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(-2+L_2)}{2L_1^2} + \frac{L_2(6 - 9L_2 + 2L_2^2)}{6L_1^3} + & \\ \quad + \frac{L_2(-12 + 36L - 2 - 22L_2^2 + 3L_2^3)}{12L_1^4} + & -0.033 < z < 0 \\ \quad + \frac{L_2(60 - 300L_2 + 350L_2^2 - 125L_2^3 + 12L_2^4)}{60L_1^5} & \end{cases} \tag{17}$$

where $\rho = -\sqrt{2(e\,z+1)}$, $L_1 = \ln(-z)$, and $L_2 = \ln[\ln(-z)]$. Formula (17) used as direct method to approximate the function $W_{-1}(z)$ for $z \in [-1/e,\,0)$ provides quite accurate values. Indeed, the maximum relative error is below 10^{-4} (see Chapeau-Blondeau and Monir, 2002).

4.3 Analytical inversion of GGD with $\alpha = 1/2$

The aim is now to invert the cumulative distribution function

$$F_X(x) = \begin{cases} \frac{1}{2}\exp\left\{-2\sqrt{30}\,(-x)^{1/2}\right\}\left(1 + 2\sqrt{30}\,(-x)^{1/2}\right) & x \le 0 \\ 1 - \exp\left\{-2\sqrt{30}\,x^{1/2}\right\}\frac{1}{2}\left(1 + 2\sqrt{30}\,x^{1/2}\right) & x > 0\,. \end{cases} \tag{18}$$

and prove that this inverse can be expressed in term of the Lambert W function. We first consider the distribution (18) for $x \le 0$. Setting $y = -(-2\sqrt{30}x)^{1/2}$, we have

$$F = \frac{1}{2}(1-y)e^y \tag{19}$$

i.e.

$$-\frac{2\,F}{e} = (1-y)e^{y-1}. \tag{20}$$

Equation (20) leads to

$$y - 1 = W_{-1}(-2F/e) \tag{21}$$

and therefore

$$x = -\frac{1}{2\sqrt{30}}[1 + W_{-1}(-2F/e)]^2. \tag{22}$$

The correct branch W_{-1} is identified from the boundary conditions ($x = 0$, $F = 1/2$) and ($x \to -\infty$, $F \to 1/2$).

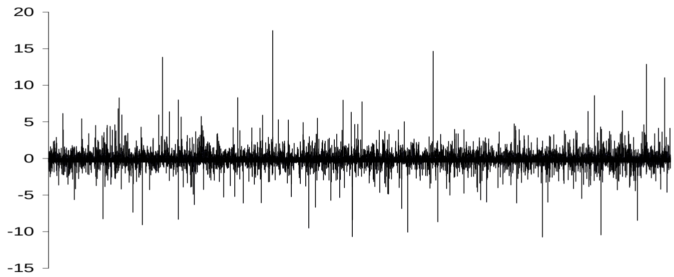


Figure 6: Instances of GGD with parameter $\alpha = 1/2$, $\mu = 0$, and $\sigma = 1$ ($N = 10\,000$).

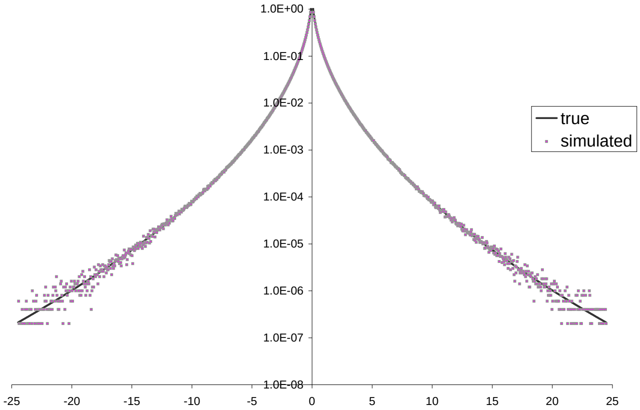


Figure 7: Estimated and theoretical probability density functions of a generalized Gaussian random variable with $\alpha = 1/2$.

Similar arguments permit to consider the case $x \geq 0$. Thus, we have the inverse cumulative distribution function

$$F^{-1}(u) = \begin{cases} -\dfrac{1}{2\sqrt{30}}\left[1 + W_{-1}(-2u/e)\right]^2 & 0 < u \leq 1/2 \\ \\ -\dfrac{1}{2\sqrt{30}}\left[1 + W_{-1}(-2(1-u)/e)\right]^2 & 1/2 < u < 1\,. \end{cases} \tag{23}$$

Equation (23) solves the problem of obtaining a large number of realizations of generalized Gaussian density with $\alpha = 1/2$, provided we are able to compute the function W_{-1}. Based on such results, a fast and fairly accurate simulation procedure can be defined: one need generate a random number uniform on (0, 1), then approximate the function W, and finally calculate F^{-1}.

4.4 Numerical results ($\alpha = 1/2$)

Realizations of a generalized Gaussian random variable with $\alpha = 1/2$, generated with the algorithm based on the inverse cumulative distribution function (23) and the approximation of the Lambert W are shown in figure 6.

We have performed a Monte Carlo estimation of probability density (13) based on 10^8 values, collected into bins of width $\Delta x = 0.05$. Figure 7 shows the simulated probability density function and the theoretical density (12). As can be seen in the figure, the probability density function is well approximated by the empirical density. The Kolmogorov-Smirnov test yielded no evidence to indicate that the simulated observations were not generated from the GGD distribution.

In table 1 the estimated moments of the GGD are reported. The results are obtained on the same sample of uniform random variates (with N the number of trials). Note that, the results obtained with the method based on the inverse cumulative distribution function written in terms of the Lambert W and those obtained with the numerical inversion of F_X are the same, but the letter method is more than fifteen times slower than the previous one.

Table 2 reports the estimated moments, based on the 10^6 simulations, of the GGD and the computational time. The method based on the inverse cdf written in terms of the Lambert W function and approximation (17) is very fast. The results of the simulation based on the inverse cdf are not reported. As already observed, such a simulation method is very inefficient with respect to the other techniques.

5 Conclusions

In this contribution, we investigated and compared different techniques for generating variates from a generalized Gaussian distribution. For $\alpha = 1/2$ a fast and accurate simulation procedure can be defined. A three-step procedure based on a gamma transformation can also be successfully applied.

Generalized Gaussian distributions have been proposed in many applications in science and in engineering. It seems interesting to investigate their possible applications in modeling economic and financial data. This is left for future research.

Table 1: Simulating the generalized Gaussian distribution with $\alpha = 1/2$ (kurtosis $\mathcal{K} = 25.2$).

	Lambert W & approximation (17)		
N	mean	variance	kurtosis
1 000	-0.004178	1.547446	40.866983
10 000	0.003823	1.004979	19.623124
100 000	0.004651	1.002315	28.279586
1 000 000	-0.001701	1.002231	24.919660
10 000 000	-0.000295	0.999027	25.192829
	Lambert W & Halley's algorithm with initial value (17)		
N	mean	variance	kurtosis
1 000	-0.004178	1.547457	40.866199
10 000	0.003823	1.004996	19.622345
100 000	0.004651	1.002332	28.278489
1 000 000	-0.001702	1.002250	24.918655
10 000 000	-0.000295	0.999045	25.191815
	Numerical inversion of F_X		
N	mean	variance	kurtosis
1 000	-0.004178	1.547457	40.866198
10 000	0.003823	1.004996	19.622345
100 000	0.004651	1.002332	28.278488
1 000 000	-0.001702	1.002250	24.918654
10 000 000	-0.000295	0.999045	25.191815

Table 2: Simulating the generalized Gaussian distribution with $\alpha = 1/2$ $(N = 10^6)$.

simulation method	mean	variance	kurtosis $\mathcal{K} = 25.2$	cpu time (sec.)
Lambert W & approximation (17)	-0.000101	0.999951	25.396091	1.97
Lambert W & Halley's algorithm	-0.000101	0.999969	25.395066	4.10
three-step procedure	-0.000077	1.000630	25.103438	10.77

Appendix A. Generalized Gaussian random variable as a transformation of a gamma random variable

Consider the case in which the random variable of interest X has a generalized Gaussian distribution with parameter α, $X \sim GG(\alpha)$. X is an absolutely continuous random variable. Let $f_X(\cdot)$ and $F_X(\cdot)$ denote the probability density function (pdf) and the cumulative distribution function (cdf) of X respectively. We have

$$f_X(x;\, \alpha) = \frac{\alpha}{2} \frac{A(\alpha)}{\Gamma(1/\alpha)} \exp\left\{-\left(A(\alpha)\,|\,x\,|\,\right)^{\alpha}\right\} \qquad x \in \mathbb{R}\,, \tag{24}$$

where

$$A(\alpha) = \left[\frac{\Gamma(3/\alpha)}{\Gamma(1/\alpha)}\right]^{1/2} \tag{25}$$

and $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$ $(z > 0)$.

The cdf of X is given by

$$F_X(x;\, \alpha) = \int_{-\infty}^{x} \frac{\alpha}{2} \frac{A(\alpha)}{\Gamma(1/\alpha)} \exp\left\{-\left(A(\alpha)\,|\,t\,|\,\right)^{\alpha}\right\} dt\,. \tag{26}$$

As already noticed, such a function can be written in closed form only in a very few special cases.

Suppose we are able to simulate the random variable Y, which is a transformation of X. In particular, we consider the relationship

$$Y = h(X) = |\,X\,|\,. \tag{27}$$

The aim is to simulate X given random observations of Y.

For a specific random generated value y_i, there exist two distinct roots of (27), $x_1^i = -y_i$ and $x_2^i = y_i$. The main problem is how to determine the probability for choosing one of the roots. According to a result of Michael *et al.* (1976)[5], it can be shown that the probability of the root x_j^i given y_i, denoted by $p_j(y_i)$, satisfies[6]

$$p_j(y_i) = \left\{ 1 + \sum_{\substack{k=1 \\ k \neq j}}^{2} \left| \frac{h'(x_l^i)}{h'(x_j^i)} \right| \frac{f_X(x_l^i)}{f_X(x_j^i)} \right\}^{-1} \qquad j = 1, 2\,. \tag{28}$$

Since the distribution of a generalized Gaussian random variable is symmetric, $f_X(x_1^i) = f_X(x_2^i)$ and being $|\, h'(x_1^i)\,| = |\, h'(x_2^i)\,|$, then $p_1(y_i) = p_2(y_i) = \frac{1}{2}$.

Let $f_Y(\cdot)$ denote the pdf of Y, then

$$f_Y(y;\,\alpha) = \frac{\alpha\, A(\alpha)}{\Gamma(1/\alpha)} \exp\left\{-\left(A(\alpha)\, y\,\right)^{\alpha}\right\} \qquad y \geq 0\,. \tag{29}$$

Consider now a random variable Z with pdf

$$f_Z(z;\, a, b) = \frac{b^a}{\Gamma(a)} z^{a-1} e^{-bz} \qquad z > 0,\ a > 0,\ b > 0\,. \tag{30}$$

Z is gamma distributed with parameters a and b, $Z \sim Gamma(a, b)$.

If we take $a = \alpha^{-1}$ and $b = A(\alpha)^{\alpha}$, and consider the transformation $Y = Z^{1/\alpha}$, it can be shown that Y has the distribution of an absolute value generalized Gaussian random variable with pdf as in (29).

We have $z = y^{\alpha}$ and $dz/dy = \alpha y^{\alpha-1}$. By substitution in (30), and considering the cdf of Z, we obtain

$$\begin{aligned} F_Z(z) &= \int_0^z \frac{b^a}{\Gamma(a)} t^{a-1} e^{-b\,t}\, dt \\ &= \int_0^z \frac{A(\alpha)}{\Gamma(1/\alpha)} t^{1/\alpha - 1} e^{-A(\alpha)^{\alpha}\, t}\, dt \\ &= \int_0^y \frac{A(\alpha)}{\Gamma(1/\alpha)}\, u^{1-\alpha} e^{-A(\alpha)^{\alpha} u^{\alpha}}\, \alpha\, u^{\alpha-1}\, du \\ &= \int_0^y \frac{\alpha A(\alpha)}{\Gamma(1/\alpha)} e^{-(A(\alpha)\, u)^{\alpha}}\, du\,. \end{aligned} \tag{31}$$

Finally, considering transformation (27), we have

$$\begin{aligned} F_Y(y) &= \int_0^y \frac{\alpha A(\alpha)}{\Gamma(1/\alpha)} e^{-(A(\alpha)\, u)^{\alpha}}\, du \\ &= \int_{-\infty}^{x} \frac{\alpha}{2} \frac{A(\alpha)}{\Gamma(1/\alpha)} e^{-(A(\alpha)|\,\xi\,|)^{\alpha}}\, d\xi\,. \end{aligned} \tag{32}$$

[5] We are grateful to Kostas Kokkinakis for providing us this reference.

[6] The result holds under more general assumptions, letting X and Y be absolutely continuous random variables and $h(\cdot)$ a function such that its first derivative exists, is continuous and is nonzero except on a null probability set. Such a function may have multiple roots.

Therefore X has a generalized Gaussian distribution with parameter α. We have now justified generating GG variates from gamma variates.

To obtain values of a GG random variable X with parameter α, the following three-step procedure can be used[7]:

i) simulate a gamma random variable Z with pdf as in (30), z_i for $i = 1, 2, \dots, n$;

ii) calculate $y_i = z_i^{1/\alpha}$, which are random variates from the distribution of Y as in (29);

iii) take $X = (-1)^B Y$, where B is a Bernoulli random variable with parameter 1/2.

Appendix B. Simulating a gamma random variable

To apply the three-step procedure described in section 3 and in Appendix A, first of all, one need simulate a gamma distribution. In the numerical experiments we used the routine DRNGAM of the IMSL library for the generation of deviates from a distribution $Gamma(a, 1)$. Variates from a $Gamma(a, b)$ distribution can be obtained by exploiting a scaling property of the gamma distribution: if $Z \sim Gamma(a, b)$, then $b\,Z \sim Gamma(a, 1)$. Hence, one obtains variates from a Gamma(a,b) simply by dividing the results by b.

The problem is how to simulate a $Gamma(a, 1)$ random variable. We describe a simulation procedure which is based on an acceptance-rejection method (for more details see Glasserman, 2004).

It is well known that, if $a = 1$ and $b = 1$, the $Gamma(1, 1)$ distribution corresponds to an exponential distribution with parameter 1, $Z \sim Exp(1)$. In order to simulate an exponential random variable, one can generate variates from a uniform distribution and take the transformation $Z = -\ln(U)$ where $U \sim \mathcal{U}[0, 1]$. Then $Z \sim Gamma(1, 1)$.

We also have that, if $Z = \sum_{k=1}^{N} - \ln U_k$, where U_k are independent and identically distributed random variables uniform on $[0, 1]$, then $Z \sim Gamma(N, 1)$. This solves the problem of sampling from a Gamma distribution with a integer.

Consider the problem of simulating a $Gamma(\theta, 1)$, with $0 < \theta < 1$. Let $\nu = \frac{e}{e+\theta}$, with $e = \exp(1)$. The following procedure can bu used:

1. generate v_1 and v_2 from two independent random variables V_1 and V_2, uniform on $[0, 1]$;
2. if $v_1 \leq \nu$, then calculate $\xi = \left(\frac{v_1}{\nu}\right)^{\frac{1}{\theta}}$ and $\eta = v_2 \xi^{\theta-1}$, else calculate $\xi = 1 - \ln\left(\frac{v_1-\nu}{1-\nu}\right)$ and $\eta = v_2 e^{-\xi}$;
3. if $\eta > \xi^{\theta-1} e^{-\xi}$, go back to step 1;
4. take ξ as a realization of a $Gamma(\theta, 1)$ random variable.

[7] See also Domínguez-Molina *et al.* (2001).

We can now deal with the general case. Let $\theta = a - \lfloor a \rfloor$, where $\lfloor a \rfloor$ is the integer part of a (hence θ is the fractional part of a). After having obtained ξ as described above, consider

$$\zeta = \xi - \sum_{k=1}^{\lfloor a \rfloor} \ln u_k \,, \tag{33}$$

where u_k are realizations of $U_k \sim \mathcal{U}[0,1]$, which are also independent from V_1 and V_2. Finally, take ζ as a realization of a $Gamma(a,1)$ random variable.

References

[1] Alefeld, G. (1981), On the convergence of Halley's method. *The American Mathematical Monthly*, **88**(7), 530–536.

[2] Chapeau-Blondeau, F., Monir, A. (2002), Numerical evaluation of the Lambert W function and application to generation of generalized Gaussian noise with exponent 1/2. *IEEE Transactions on Signal Processing*, **50**(9), 2160–2165.

[3] Corless, R.M., Gonnet, G.H., Hare, D.E.G., Jeffrey, D.J., Knuth, D.E. (1996), On the Lambert W function. *Advances in Computational Mathematics*, **5**, 329–359.

[4] Domínguez-Molina, J.A., González-Farías, G. (2002), Sobre la estimación del parámetro de forma de la distribución gausiana generalizada. *Aportaciones Matemáticas. Serie Comunicaciones*, **30**, 117–120.

[5] Domínguez-Molina, J.A., González-Farías, G., Rodríguez-Dagnino, R.M. (2001), A practical procedure to estimate the shape parameter in the generalized Gaussian distribution. *Technique Report*, I-01-18 (`http://www.cimat.mx`).

[6] Glasserman, P. (2004), *Monte Carlo Methods in Financial Engineering*. Springer-Verlag, New York.

[7] Kokkinakis, K., Nandi, A.K. (2005), Exponent parameter estimation for a generalized Gaussian probability density functions with application to speech modeling. *Signal Processing*, **85**, 1852–1858.

[8] Michael, J.R., Schucany, W.R., Haas, R.W. (1976), Generating random variates using transformations with multiple roots. *The American Statistician*, **30**, 88–90.

[9] Miller, J.H., Thomas, J.B. (1972), Detectors for discrete-time signals in non-Gaussian noise. *IEEE Transactions of Information Theory*, **IT-18**(2), 241–250.

[10] Müller, U.A., Dacorogna, M.M., Pictet, O.V. (1998), Heavy tails in high-frequency financial data. In Adler, R.J., Feldman, R.E., Taqqu, M.S. (Eds.), *A Practical Guide to Heavy Tails. Statistical Techniques and Applications*. Birkäuser, Boston, 55–77.