The Wisdom of the Crowd in Dynamic Economies

Pietro Dindo[a] Filippo Massari[b]

[a] Ca' Foscari University of Venice
[b] University of New South Wales

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Motivation

In financial markets, asset prices are believed to be good predictors of economic performance.

Three different mechanisms:

- Prices reveal and aggregate private information: learning from prices literature (Aumann 1976, Geanakoplos-Polemarchakis 1982, Grossman 1976, Radner 1979)
- Prices reveal the beliefs of the most accurate agent: Market Selection Hypothesis literature (Friedman 1953, Blume-Easley 1992, 2006, Sandroni 2000)
- Prices balance opposite biases: Wisdom of the Crowd (Galton 1907, Surowiecki 2005) and prediction markets (Wolfers and Zitzewitz 2004, Manski 2006)

Our paper

We construct a stylized general equilibrium model of a dynamic asset market that combines the three approaches.

When agents naively incorporate equilibrium prices in their prediction:

- the equilibrium consumption-shares/belief dynamics endogenously generates a WOC
- market accuracy is a self-fulfilling prophecy

Both results disappear if agents learn in a Bayesian fashion rather than being naive.

Setting

- Arrow-Debreu exchange economy (complete markets)
- In each period $t = 1, 2, \ldots$, a finite state space $\mathcal{S}$, $|\mathcal{S}|$=S.
- $\sigma = (\sigma_1, \sigma_2, ...) \in \mathcal{S}^\infty$ is a sequence, $\sigma^t \in \mathcal{S}^t$ is a partial history till t.
- P is the true measure and we assume i.i.d. states.
- A finite set $\mathcal{I}$ of agents i with endowment $\{e^i_t(\sigma^t)\}$ and probability measure p^i.
- Agents use markets for contingent goods to choose $\{c^i_t(\sigma^t)\}$.

Assumptions A

- All agents in the economy have log utility, identical discount factor β, and solve

$$\max E_{p^i}\left[\sum_{t=0}^{\infty}\beta^t \log c_t^i(\sigma)\right] \quad s.t. \quad \sum_{t=0}^{\infty}\sum_{\sigma^t\in\mathcal{S}^t} q(\sigma^t)\left(c_t^i(\sigma^t)-e_t^i(\sigma^t)\right)\leq 0.$$

- Agent measures have common support on finite histories:

$$\forall i, \forall\sigma^t, p^i(\sigma^t)>0 \Leftrightarrow P(\sigma^t)>0.$$

- No aggregate risk: $\sum_{i\in\mathcal{I}} e_t^i(\sigma)=1$.

Market probability

Under assumptions **A** a competitive equilibrium exists.

Lemma 2.1. (Rubinstein 1974)

Under **A***, in every* σ*, the probabilities used of the representative agent are:*

$$p^M(\sigma^t) = \sum_{i \in \mathcal{I}} p^i(\sigma^t) c_0^i,$$

$$p^M(\sigma_t | \sigma^{t-1}) = \sum_{i \in \mathcal{I}} p^i(\sigma_t | \sigma^{t-1}) c_{t-1}^i(\sigma^{t-1}).$$

We name p^M market probability.

Agents' beliefs

Each agent i's next-period probabilities are the convex combination of market probabilities, p^M, and an agent-specific dogmatic probabilities π^i on S.

Definition 2.2.
Let $\alpha^i \in (0, 1]$, agent i's next-period probability are given by:

$$p^i(\sigma_t|\sigma^{t-1}) = (1 - \alpha^i)p^M(\sigma_t|\sigma^{t-1}) + \alpha^i \pi^i(\sigma_t)$$

Interpretations:

- naive learning (Manski 2006, Golub and Jackson 2010)
- anchoring effect (Shiller 1999)
- fractional Kelly rule (MacLean et al., 2011)

Note: A Bayesian learner uses Bayes rule to update α^i.

Consumption-share and belief dynamics

$$\begin{cases} p^M(\sigma_t|\sigma^{t-1}) = \sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1})c^i_{t-1}(\sigma^{t-1}), \quad \forall \sigma_t \in S. \\ \\ c^i_t(\sigma^t) = \frac{p^i(\sigma_t|\sigma^{t-1})}{p^M(\sigma_t|\sigma^{t-1})} c^i_{t-1}(\sigma^{t-1}), \quad \forall i \in \mathcal{I}, \\ \\ p^i(\sigma_t|\sigma^{t-1}) = (1-\alpha^i)p^M(\sigma_t|\sigma^{t-1}) + \alpha^i \pi^i(\sigma_t) \end{cases}$$

Survival and Accuracy

Definition 2.1.

Agent i vanishes if $\lim_{t\to\infty} c_t^i(\sigma) = 0$ *P-a.s..*
Agent i survives if $\limsup_{t\to\infty} c_t^i(\sigma) > 0$ *P-a.s..*

Definition 3.1.-3.2.

Agent i is more accurate than agent j if $D(p^i||P) < D(p^j||P)$,
where $D(p^i||P) := \lim_{t\to\infty} \frac{1}{t} \sum_1^t E_P \left[\ln \frac{P(\sigma_t|\sigma^{t-1})}{p^i(\sigma_t|\sigma^{t-1})} \right]$.

Agents survival: Sufficient conditions

Proposition 3.1. (Sandroni 2000): Under **A**,

if agent j is more accurate than agent i, then i vanishes.

Proposition 3.2. Under **A**,

(a) *no agent can be more accurate than the market;*

(b) *agent i survives only if he is as accurate as the market.*

Wisdom of the Crowd

Definition 3.3. Given a set of dogmatic probabilities $\pi^1, ...\pi^I$

- *Best Individual Probability:* $\pi^{BIP} = \arg \min_{p \in \{\pi^1, ..., \pi^I\}} D(p||P)$;
- *Best Collective Probability:* $\pi^{BCP} = \arg \min_{p \in Conv(\pi^1, ..., \pi^I)} D(p||P)$;
- *Agent dogmatic probabilities are diverse if* $\pi^{BIP} \neq \pi^{BCP}$.

Definition 3.4.

The WOC occurs if market probabilities are more accurate than the most accurate dogmatic probability:

$$D(p^M||P) < D(\pi^{BIP}||P).$$

Wisdom of the Crowd: Necessary conditions

Proposition 4.1. Under **A**,

WOC can occur only if agent beliefs depend on prices.

Proposition 4.2. Under **A**,

WOC can occur only if dogmatic probabilities are diverse.

Wisdom of the Crowd: Sufficient conditions

WOC occurs in those economies in which agents believe enough in market accuracy.

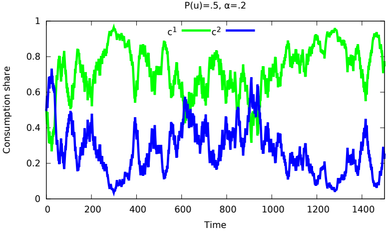
Proposition 4.3. Under **A**,

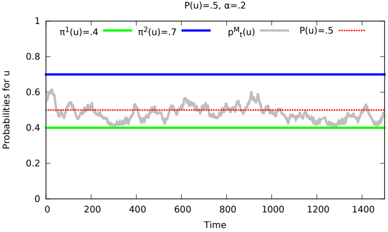
when $\pi^{BIP} \neq \pi^{BCP}$, there exists an $\bar{\alpha} \in (0,1)$ such that if $\alpha^i < \bar{\alpha}$ for all $i \in \mathcal{I}$, then WOC holds,

$$D(p^M || P) < D(\pi^{BIP} || P),$$

and at least two agents survive.

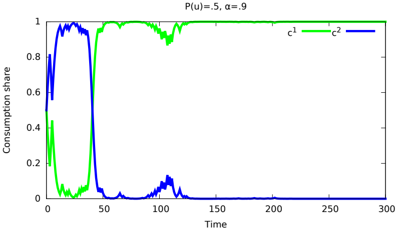
An example of WOC

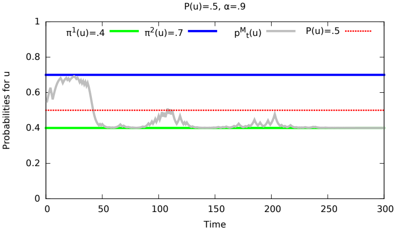




$S = \{u, d\}$	*State space*
$[c_0^1; c_0^2] = [.5; .5]$	*Consumption shares at t=0*
$P(u) = .5$	*True probability*
$[\pi(u)^1, \pi(u)^2] = [.4; .7]$	*Dogmatic Probabilities*
$P(u) = \pi^{BCP}(u) \neq \pi^{BIP}(u) = \pi^1(u)$	*Probabilities are diverse*
$[\alpha^1; \alpha^2] = [.2; .2]$	*Agents mixing coefficients are low*

An example of failure of WOC





$S = \{u, d\}$	*State space*
$[c_0^1; c_0^2] = [.5; .5]$	*Consumption shares at t=0*
$P(u) = .5$	*True probability*
$[\pi(u)^1, \pi(u)^2] = [.4; .7]$	*Dogmatic Probabilities*
$P(u) = \pi^{BCP}(u) \neq \pi^{BIP}(u) = \pi^1(u)$	*Probabilities are diverse*
$[\alpha^1; \alpha^2] = [.9; .9]$	*Agents mixing coefficients are high*

Accurate markets: A self-fulfilling prophecy

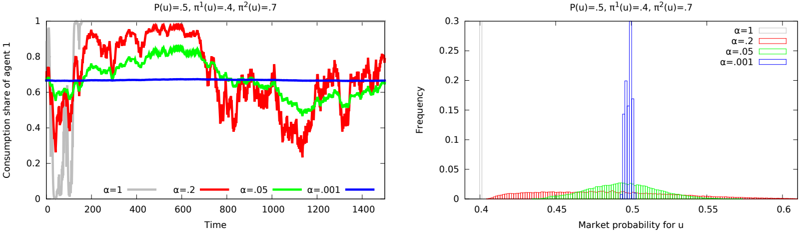
If agent dogmatic probabilities are diverse and agents believe in market accuracy, markets are indeed accurate.

Theorem 4.1.

Let $\{\mathcal{E}_\alpha\}$ *be a sequence of economies that satisfy* **A** *with market probabilities processes* $\{p^M_\alpha\}$*. All economies have two states and are identical in all respects but for the value of the mixing coefficients* α*. The following result holds:*

$$\lim_{\alpha \to 0} D(p^M_\alpha || P) = D(\pi^{BCP} || P)\,.$$

An example of accurate markets



$[\mathcal{E}_{\alpha_1}; \mathcal{E}_{\alpha_2}; \mathcal{E}_{\alpha_3}; \mathcal{E}_{\alpha_4}]$	*Four economies*
$S = \{u, d\}$	*State space*
$[c_0^1; c_0^2] = [.67; .33]$	*Consumption shares at t=0*
$P(u) = .5$	*True probability*
$[\pi(u)^1, \pi(u)^2] = [.4; .7]$	*Dogmatic Probabilities*
$P(u) = \pi^{BCP}(u) \neq \pi^{BIP}(u) = \pi^1(u)$	*Probabilities are diverse*
$[\alpha_1; \alpha_2, \alpha_3; \alpha_4] = [1; .2; .05; .001]$	*Mixing coefficients on economy [1,2,3,4]*

Survival of naive traders: the role of α^i

- Believing in market accuracy (weakly) improves agents accuracy.

Proposition 5.1. Under **A**,
for all agents $i \in \mathcal{I}$, $\forall \alpha^i \in (0,1)$, $D(p^i||P) \leq D(\pi^i||P)$.

- Believing in market accuracy provides a (weakly) monotonic evolutionary advantage.

Proposition 5.2. Under **A**,
consider two agents, i *and* j*, with* $\pi^i = \pi^j$.
If $\alpha^j < \alpha^i$ *and WOC occurs, then agent* i *vanishes.*

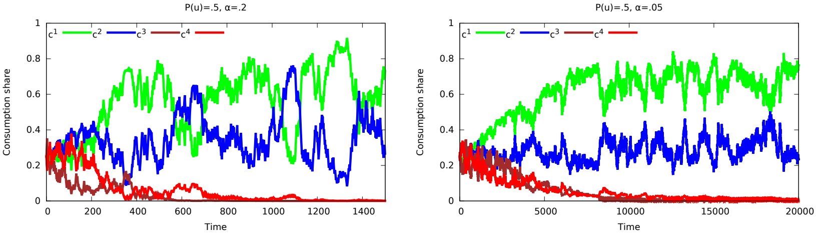
Survival of naive traders: the role of probabilities diversity

The market selects against those agents whose dogmatic probabilities are less accurate than others and cannot be used to increase market accuracy.

Corollary 5.1. Under **A**,

In an economy with two states in which all agents have the same mixing coefficient α and WOC occurs, exactly two agents survive. These are the least pessimistic and the least optimistic.

An example of traders who vanish because redundant

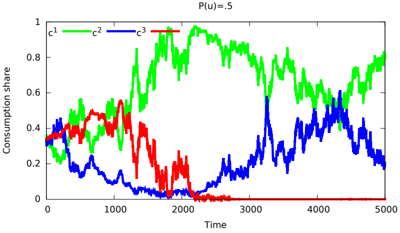


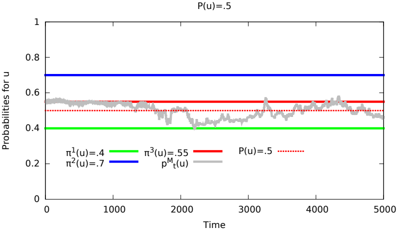
$[\mathcal{E}_{\alpha_1}; \mathcal{E}_{\alpha_2}]$	*Two economies*
$S = \{u, d\}$	*State space*
$[c_0^1; c_0^2; c_0^3; c_0^4] \approx [.25; .25; .25; .25]$	*Consumption shares at t=0*
$P(u) = .5$	*True probability*
$[\pi(u)^1, \pi(u)^2, \pi(u)^3, \pi(u)^4] = [.4; .7; .2; .8]$	*Dogmatic probabilities*
$P(u) = \pi^{BCP}(u) \neq \pi^{BIP}(u)$	*Probabilities are diverse*
$[\alpha_1; \alpha_2] = [.2; .05]$	*Mixing coefficients of economy [1,2]*

Importance of mixing

If other traders use market probabilities, you should do the same.

Corollary 5.2. Under **A**,
under WOC, when $\pi^i \neq p^{BCP}$ $\exists \alpha^* \in (0,1)$ *such that if* $\alpha^j < \alpha^*$ $\forall j \in \mathcal{I} \setminus i$ *and* $\alpha^i = 0$*, then trader* i *vanishes* P*-a.s..*





No WOC under Bayesian Learning

Definition 5.1.

For all $i \in \mathcal{I}$, agent i beliefs are given by

$$p^i(\sigma_t|\sigma^{t-1}) = \left(1 - \alpha^i_{t-1}(\sigma^{t-1})\right) p^M(\sigma_t|\sigma^{t-1}) + \alpha^i_{t-1}(\sigma^{t-1})\pi^i(\sigma_t)$$

where $\alpha^i_0 \in (0,1)$ *and* $\alpha^i_{t-1}(\sigma^{t-1}) := \frac{\alpha_0\pi^i(\sigma^{t-1})}{(1-\alpha_0)p^M(\sigma^{t-1})+\alpha_0\pi^i(\sigma^{t-1})}$.

Proposition 5.4. Under **A**,

if agents weights α^i_t are updated using Bayes rule above, then

(a) *all agents survive on every sequence σ;*

(b) *no WOC occurs and* $\lim_{t\to\infty} p^M(\cdot|\sigma^{t-1}) = \pi^{BIP}(\cdot)$, *P-a.s..*

Conclusions

- We construct a dynamic GE model with in which MSH and WOC coexist.
- We show that market accuracy is a self-fulfilling prophecy.
- We show that both results disappear if agents learn in a Bayesian fashion rather than being naive.